Chapter 2

Preliminaries

In this chapter we collect some of the basic facts from algebra, topology, functional analysis and the theory of topological groups. This chapter cannot replace full courses in these topics and it is not expected that a reader unfamiliar with these fields can master them on first reading. We include this short summary to make these notes more or less self contained and to give the reader a feeling for the problems investigated in these branches of mathematics. In this chapter we will also introduce some basic notations and some important results for easier reference. On first reading the reader may browse through this chapter and come back to it later when reading the following chapters. For a more detailed treatment of these topics it is necessary to consult the standard literature.

2.1 Topology and Functional Analysis

We introduce first some basic notations and generalize then the concept of a neighborhood of a point by introducing topological spaces. Such a space is a set together with a system of subsets of the original set. The subsets in this system satisfy certain conditions that characterize open subsets in the usual meaning. We call the subsets in this system open subsets. We will not develop topology in any detail but we will only introduce the concepts of compact and locally compact sets and continuous mappings between topological spaces.

Then we give a short overview of Lebesgue integration that will be used later when we introduce invariant integration over groups. Finally we introduce some basic definitions from functional analysis, especially Hilbert spaces and operators between them.

This section collects only the most important definitions and some results from these areas and the interested reader may wish to consult the standard literature for further information.

2.1.1 Some Notations

By $\emptyset$ we denote the *empty set*, by $\{x, y, z\}$ the set with the elements x, y and z and by $\{x|c(x)\}$ the set of all elements which satisfy the condition $c(x)$.

The notation $f : A \rightarrow B; a \mapsto b = f(a)$ describes the *function* f that maps the set A into the set B. f maps the element $a \in A$ into the element $b = f(a)$. If $X \subset B$ then we denote by $f^{-1}(X)$ the set

$$f^{-1}(X) = \{a \in A | f(a) \in X\} \tag{2.1}$$

By $f(A)$ or $im(f)$ we denote the set of all elements in A that are mapped into X under f:

$$f(A) = \{b \in B | b = f(a) \text{ for some } a \in A\}. \tag{2.2}$$

We call $im(f)$ the *image* of A under f. If $f : A \to B$ and $g : B \to C$ then we denote by $g \circ f$ the *composition* of f and g defined as:

$$g \circ f : A \to C; a \mapsto g(f(a)). \tag{2.3}$$

A map is called *injective* if $f(a) = f(b)$ implies $a = b$. An injective function maps different elements in A to different elements in B. A map is called *surjective* if for all $b \in B$ there is an element $a \in A$ such that $f(a) = b$, or $im(f) = B$. A map is called *bijective* if it is injective and surjective.

If E_α is a collection of sets, where α runs over some index set A, then we define by $\bigcup_{\{\alpha \in A\}} E_\alpha$ the *union* of all the sets E_α and by $\bigcap_{\{\alpha \in A\}} E_\alpha$ their *intersection*. If A and B are two sets then we define their *product* $A \times B$ as the set $\{(a, b) | a \in A, b \in B\}$. The *Kronecker symbol* δ_{ij} is defined as the function which has the value 1 if $i = j$ and 0 if $i \neq j$.

Finally we recall the notation of an *equivalence relation*. Assume A is a set. Then we say that the subset X of $A \times A$ defines a equivalence relation if the following conditions are satisfied:

- $(a, a) \in X$ for all $a \in A$
- If $(a, b) \in X$ and $(b, c) \in X$ then $(a, c) \in X$
- If $(a, b) \in X$ then $(b, a) \in X$

If X defines an equivalence relation and $(a, b) \in X$ then we write $a \equiv b$. The set $E(a) = \{b \in A | a \equiv b\}$ is called an *equivalence class* and it is easy to show that an equivalence relation partitions A in the sense that $\bigcup_{a \in A} E(a) = A$ and $E(a) \bigcap E(b) = E(a)$ or $E(a) \bigcap E(b) = \emptyset$.

2.1.2 Topological Spaces

In the first definition we generalize the concept of open sets for arbitrary spaces.

Definition 2.1 1. A set X together with a family $\mathcal{U} = \{U\}$ of subsets $U \subset X$ is called a *topological space* if it satisfies the following conditions:

- $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$.
- The union of any family of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.
- The intersection of any finite number of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

2. The elements of $\mathcal{U}$ are called the *open sets* of the topological space X.

3. The family $\mathcal{U}$ is called the *topology* of X.
4. A set $\mathcal{V} = \{V\}$ of elements $V \in \mathcal{U}$ is called a *basis for the topology* $\mathcal{U}$ if every element $U \in \mathcal{U}$ is the union of certain elements $V \in \mathcal{V}$.

We note that a topological space consists of two elements, the set X and the topology $\mathcal{U}$. A topological space should therefore be denoted by the pair $(X,\mathcal{U})$, we will however mainly speak of the topological space X assuming that the topology is clear from the context. We have to denote the topology explicitly if we consider two different topologies on the same set X.

Examples 2.1 1. Take as $X = \mathbf{R}$ the set of real numbers. If we take the open intervals as the basis of the topology then we get the usual topology on $\mathbf{R}$.

2. Take any set X and define $\mathcal{U} = \{\emptyset, X\}$. This is the coarsest topology on X.
3. Take any set X and define $\mathcal{U}$ as the set of all subsets of X. This is the finest topology on X. It is called the *discrete topology.*

With the help of a topology it is possible to generalize the concept of a neighborhood of a point:

Definition 2.2 If $x \in X$ is a point in X, then we call any open set $U \in \mathcal{U}$ containing x a *neighborhood of the point* x. Neighborhoods of $x \in X$ will often be denoted by $U(x)$.

With this definition we find easily the familiar characterization of an open set:

Theorem 2.1 A subset $M \subset X$ is open if and only if every point $x \in M$ has a neighborhood $U(x)$ such that $U(x) \subset M$.

Note that the last statement is a theorem and not the definition of an open set. One construction which will be frequently used in the following is the definition of a topology on a subset $Y \subset X$. We define:

Definition 2.3 Assume $(X,\mathcal{U})$ is a topological space and $Y \subset X$. Then we define a topology $\mathcal{U}'$ on Y by using the intersections $Y \bigcap U$ as open sets on Y:

$$\mathcal{U}' = \left\{ Y \bigcap U | U \in \mathcal{U} \right\}.$$

Closed sets are as usual defined as the complements of the open sets:

Definition 2.4 1. A subset $C \subset X$ is called a *closed subset* if the complement $X - C$ is open, i.e. $X - C \in \mathcal{U}$.

2. The intersection of all closed subsets containing $M \subset X$ is called the *closure of* M. It is denoted by $\overline{M}$.
3. An element in $\overline{M}$ is called a *limit point* of M.

The definition of a compact set is motivated by the well-known Heine-Borel theorem:

Definition 2.5 1. A family $\{O\}$ of subsets of X is called a *cover* if the union is equal to X : $X = \bigcup O$. If all sets O are open then we call it an open cover.

2. A topological space X is called *compact* if every open cover $\{O\}$ of X contains a finite cover $\{O_1, ..., O_n\}$.
3. A subset $Y \subset X$ is called *compact* if it is compact when regarded as a subspace of X.
4. A space X is called *locally compact* if every point $x \in X$ has a neighborhood whose closure is compact.
5. Assume $x_1, ..., x_n, ...$ is a sequence of elements in X. The point $x \in X$ is called a *limit of the sequence* if for every neighborhood $U(x)$ of x, there is a positive integer n_0 such that all elements $x_{n_0}, x_{n_0+1}, ...$ of the sequence belong to $U(x)$. We write $x = \lim_{n\to\infty} x_n$.

As a last type of topological spaces we introduce separated or Hausdorff spaces since we will always assume that the topological spaces are separated.

Definition 2.6 A topological space X is called *separated* or a *Hausdorff space* if every two points $x_1, x_2 \in X$ can be separated by disjoint neighborhoods. For every two points $x_1 \neq x_2$ we can thus find two neighborhoods $U(x_1), U(x_2)$ such that $U(x_1) \bigcap U(x_2) = \emptyset$.

Finally we will now define continuous mappings between arbitrary topological spaces:

Definition 2.7 Assume X and Y are two topological spaces and $f : X \to Y$ is a mapping.

1. Assume x_0 is a point in X and $y_0 = f(x_0)$. f is called *continuous at the point* $x_0 \in X$ if the inverse image $f^{-1}(V(y_0))$ of every neighborhood $V(y_0)$ contains some neighborhood $U(x_0)$ of x_0 : $U(x_0) \subset f^{-1}(V(y_0))$.
2. f is called *continuous* if f is continuous on every point in X.
3. f is called a *homeomorphism* if:
   - f is bijective and if
   - f and f^{-1} are continuous

For continuous mappings we find the following characterization:

Theorem 2.2 1. If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are topological spaces and f is a function $f : X \to Y$, then f is continuous if and only if the inverse image of every open set in Y is open in X: $f^{-1}(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$.

2. If f is a function $f : X \to Y$, then f is continuous if and only if the inverse image of every closed set in Y is closed in X.
3. Under a homeomorphism open sets are mapped onto open sets and closed sets are mapped onto closed sets.

2.1.3 Lebesgue Integrals

We summarize first some basic facts about Lebesgue integration since the existence of a special (group-invariant) integral is an essential result needed in the study of so-called locally-compact groups (see section 2.3.3). For a detailed treatment of measure and integration theory the reader may consult the literature (for example [17]).

Definition 2.8 1. Assume $\mathcal{S}^n$ is the set of all bounded intervals in $\mathbf{R}^n$. Then we say that a function ϕ is an *interval function* if $\phi : \mathcal{S}^n \to \mathbf{R}$.

2. An interval function ϕ is:
   - *monotone* if for all $I_1, I_2 \in \mathcal{S}^n$ with $I_1 \subset I_2 : \phi(I_1) \leq \phi(I_2)$.
   - *additive* if for all $I_1, I_2, I \in \mathcal{S}^n$ with $I_1 \bigcap I_2 = \emptyset$ and $I_1 \bigcup I_2 = I : \phi(I) = \phi(I_1) + \phi(I_2)$
   - *regular* if for all $\epsilon > 0$ and all $I \in \mathcal{S}^n$ there is an open interval $I^* \supset I$ such that $\phi(I) \leq \phi(I^*) < \phi(I) + \epsilon$.

3. A *measure* μ is a monotone, additive and regular interval function.

A typical interval function is the length (or the volume) of an interval: $\phi(I) = \phi([a,b]) = b - a$.

We use a measure on $\mathbf{R}^n$ to define the integral over those simple functions that are constant on intervals and have non-zero value only on a finite number of intervals:

Definition 2.9 Assume $I_1, ..., I_n$ is a finite number of intervals in $\mathbf{R}^n$ and μ is a measure. Assume further that $f : \mathbf{R}^n \to \mathbf{R}$ is a function that is constant on I_j where it has the value $c_j = f(I_j)$. Assume further that it is zero outside these intervals. Then we define the *Lebesgue integral* as $\int f \, d\mu = \int_{\mathbf{R}^n} f \, d\mu = \sum_{j=1}^n c_j \mu(I_j)$.

With the help of a limit operation this integral is then extended to a larger class of functions:

First we define $C_1(\mathbf{R}^n, \mu)$ as the set of functions for which there is a monotonically increasing sequence of interval functions f_n such that $f = \lim f_n$ and $\int f_n \, d\mu \leq A$ for a constant A and all indices n. For the functions $f \in C_1(\mathbf{R}^n, \mu)$ we define $\int f \, d\mu = \lim \int f_n \, d\mu$. Then we define $C_2(\mathbf{R}^n, \mu) = \{f_1 - f_2 | f_i \in C_1(\mathbf{R}^n, \mu)\}$ and set $\int f \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu$ for all $f \in C_2(\mathbf{R}^n, \mu)$.

Definition 2.10 A function f is called *integrable* if it is an element of $C_2(\mathbf{R}^n, \mu) : \quad f \in C_2(\mathbf{R}^n, \mu)$.

To be correct one would have to change the previous definitions in such a way that two functions which have different values only on a set of measure zero are considered to be equal.

One of the main reasons for introducing a new integral (which is a generalization of the Riemann integral) is the ability to interchange integration and limit operations. One example of such a property and one of the main results of the theory is the following theorem:

Theorem 2.3 [Lebesgue] Assume f_n is a series of integrable functions and $f = \lim f_n$. Assume further that there is an integrable function g such that $|f_n| \leq g$ for all n. Then f is integrable and

$$\int \lim f_n \, d\mu = \int f \, d\mu = \lim \int f_n \, d\mu.$$

Finally we introduce the concept of a measurable function:

Definition 2.11 A function $f : \mathbf{R}^n \to \mathbf{R}$ is called *measurable* if there is a sequence of interval functions f_n such that the $\lim f_n = f$.

Note that measurable functions are not necessarily integrable since we required in the definition of the set C_1 that the integrals over the interval functions were bounded by a fixed constant. The definitions of measurable and integrable functions can easily be generalized to the case where the functions are complex valued. In this case one treats the function as a sum of two real functions $f = f_r + i \cdot f_i$ and defines $\int f \, d\mu = \int f_r \, d\mu + i \int f_i \, d\mu$.

With the help of the previous definitions we can introduce the spaces L^p defined as follows:

Definition 2.12 If $p > 0$ then:

1. $L^p(\mathbf{R}^n, \mu) = \{f : \mathbf{R}^n \to \mathbf{R} | f \text{ is measurable and} |f|^p \text{ is integrable}\}$
2. $L^p_{\mathbf{C}}(\mathbf{R}^n, \mu) = \{f : \mathbf{R}^n \to \mathbf{C} | f \text{ is measurable and} |f|^p \text{ is integrable}\}$

2.1.4 Functional Analysis

In this section we summarize some important concepts from functional analysis. This is only offered as a collection of results and not as an introduction to functional analysis. For such an introduction the reader is referred to the literature (see for example [41], [19], [53] and [10]).

We first introduce a norm as a map that generalizes the concept of length from geometry. Then we introduce a scalar product which is used to define the length of elements in a vector space and the angle between two elements in a vector space.

Definition 2.13 1. A *real vector space* is a set X together with two mappings $+ : X \times X \to X$ and $\cdot : \mathbf{R} \times X \to X$ such that the following conditions are satisfied for all $x, y, z \in X$ and all $k, k_1, k_2 \in \mathbf{R}$:

(a) $(x + y) + z = x + (y + z)$

(b) There is a zero element $0 : x + 0 = x$ for all $x \in X$.

(c) For all $x \in X$ there is an inverse element $-x \in X : x - x = 0$.

(d) $x + y = y + x$

(e) $(k_1 + k_2) \cdot x = k_1 \cdot x + k_2 \cdot x$

(f) $k \cdot (x + y) = k \cdot x + k \cdot y$

(g) $(k_1 k_2) \cdot x = k_1 \cdot (k_2 \cdot x)$

(h) $1 \cdot x = x$

2. A set X is called a *complex vector space* if $\cdot$ is a mapping $\cdot : \mathbf{C} \times X \to X$ and if the conditions (5-7) in the previous definition hold for all $k, k_1, k_2 \in \mathbf{C}$.
3. An element x of a vector space X is called a *vector*.

In this section we will normally only speak of a vector space, understanding that it can be either a real or a complex vector space. If a result or a definition holds only for either real or complex spaces then we will mention it explicitly. Instead of $k \cdot x$ we will also write $kx = k \cdot x$. In the next definition we introduce the norm of an element in a vector space.

Definition 2.14 Assume X is a vector space.

1. A mapping $\|\| : X \to \mathbf{R}$ is called a *norm* if it satisfies the following conditions:
   - $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,
   - $\|kx\| = |k|\|x\|$ for all constants k and all vectors x
   - $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (Triangle inequality)
2. A *normed space* is a vector space with a norm.

Normed spaces may contain holes and we define therefore complete spaces as spaces that contain all limit points:

Definition 2.15 1. A sequence of vectors $(x_n)_{n\in\mathbf{Z}}$ of a normed space X forms a *Cauchy sequence* if it satisfies the following condition: For all $\epsilon > 0$ there is an index n_0 such that for all $n, m > n_0 : \|x_n - x_m\| < \epsilon$.

2. A normed space X is called *complete* or a *Banach space* if every Cauchy sequence $(x_n)_{n\in\mathbf{Z}}$ converges to an element $x \in X$.

The norm $\|x_n - x_m\|$ measures the distance between the elements x_n and x_m and a Cauchy sequence is thus a sequence whose elements lie eventually arbitrary near each other. One important class of Banach spaces is that of L^p spaces:

Theorem 2.4 [Riesz-Fischer] Define on $L^p(\mathbf{R}^n, \mu)$: $\|f\| = \left(\int |f|^p \, d\mu\right)^{1/p}$. This defines a norm and the spaces $L^p(\mathbf{R}^n, \mu)$ are complete for all $p \geq 1$.

A special type of vector spaces are the vector spaces that possess a scalar product. In these spaces we can measure the length of vectors and angles between vectors:

Definition 2.16 Assume X is a complex vector space. A *scalar product* $\langle \, , \, \rangle$ is a mapping on $X \times X$ which satisfies the following conditions for all $x, y, z \in X$ and all $c \in \mathbf{C}$:

- $\langle \, , \, \rangle : X \times X \to \mathbf{C}$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c \langle x, y \rangle$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

($\overline{x}$ is the conjugate complex of x.).

If X is a real vector space then we define a scalar product as a mapping $\langle\ ,\ \rangle : X \times X \to \mathbf{R}$ with the properties given in definition 2.16. Note that we have $\langle x, y\rangle = \langle y, x\rangle$.

Examples are the space $L^2(\mathbf{R}^n, \mu)$ with the scalar product defined as:

$$\langle f, g\rangle = \int fg \, d\mu.$$

and the complex vector space $L^2_{\mathbf{C}}(\mathbf{R}^n, \mu)$ with:

$$\langle f, g\rangle = \int f\overline{g} \, d\mu.$$

On the n-dimensional real and complex spaces $\mathbf{R}^n$ and $\mathbf{C}^n$ we define as usual $\langle x, y\rangle = x'y$ and $\langle x, y\rangle = x'\overline{y}$ where x' is the transposed vector x and $x'y$ is the inner product of the two vectors x and y.

In the spaces $\mathbf{R}^n$ with the common scalar product it is well known that the scalar product of two vectors is proportional to the cosine of the angle between the vectors. We define therefore in the general case:

Definition 2.17 Assume X is a vector space with a scalar product. Then we define:

1. Two elements $x, y \in X$ are said to be *orthogonal* if $\langle x, y\rangle = 0$.
2. Two subsets $M_1, M_2 \subset X$ are said to be *orthogonal* if $\langle x, y\rangle = 0$ for all $x \in M_1$ and all $y \in M_2$.
3. Given that $M \subset X$ is an arbitrary subset of X we define the *orthogonal complement* $M^\perp$ as:
$$M^\perp = \{y \in X \mid \langle x, y\rangle = 0 \text{ for all } x \in M\}$$
4. A subset $\{x_i\}$ is called *orthonormal* if $\langle x_i, x_j\rangle = \delta_{ij}$ for all i, j.

For vector spaces with scalar products one gets the following two theorems:

Theorem 2.5 [Cauchy-Schwarz-Inequality] Assume X is a vector space with scalar product $\langle\ ,\ \rangle$, then $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$ for all $x, y \in X$.

Theorem 2.6 Assume X is a vector space with scalar product then $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on X.

Since a vector space with a scalar product is also a normed space it is meaningful to consider those spaces that are complete in this norm:

Definition 2.18 Assume H is a vector space with scalar product $\langle\ ,\ \rangle$ and $\|x\|$ is the norm defined in theorem 2.6. Then we say that H is a *Hilbert space* if H is complete in this norm.

For our purposes the most important examples of Hilbert spaces are the $L^2(\mathbf{R}^n, \mu)$ spaces.

The elements in Hilbert spaces can be decomposed into a series of basis elements which makes the structure of these Hilbert spaces especially simple. We define:

Definition 2.19 Assume H is a Hilbert space.

1. The orthonormal subset $\{x_i\}$ is ***complete*** or ***maximal*** if the following condition holds: Assume $\{y_i\}$ is another orthonormal subset of H that contains all vectors x_i. Then the two orthonormal systems $\{x_i\}$ and $\{y_i\}$ are equal as sets.
2. A complete orthonormal subset of H is called a ***basis*** of H.

For orthonormal subsets of a Hilbert space we have the following inequality:

Theorem 2.7 [Bessel Inequality] Assume $\{x_i\}$ is an orthonormal subset of a Hilbert space H. Then we have the following inequality:

$$\sum \langle x, x_i \rangle^2 \leq \|x\|^2$$

for all $x \in H$.

In the following theorem we collect some conditions which describe the structure of a Hilbert space completely:

Theorem 2.8 Assume H is a Hilbert space and $\{x_i\}$ is an orthonormal subset of H. Then the following statements are equivalent:

- $\{x_i\}$ is complete
- If $\langle x, x_i \rangle = 0$ for all x_i then $x = 0$
- For all $x \in H$ we have the *Fourier decomposition:* $x = \sum \langle x, x_i \rangle x_i$
- For all $x, y \in H$ we have: $\langle x, y \rangle = \sum \langle x, x_i \rangle \langle y, x_i \rangle$.
- For all $x \in H$ we have the *Parseval equality:* $\|x\|^2 = \sum |\langle x, x_i \rangle|^2$.

As in the case of finite-dimensional vector spaces we find therefore that the Hilbert space is the set of all linear combinations of a basis.

We conclude this section on functional analysis with some basic facts about linear functions on Hilbert spaces:

Theorem 2.9 Assume X, Y are two normed spaces and $T : X \rightarrow Y$ is a linear operator. Then the we have:

1. The following three conditions are equivalent:
   - T is continuous
   - T is continuous at 0
   - T is continuous at every point $x \in X$.
2. T is continuous if and only if there is a constant $A \geq 0$ such that $\|T(x)\| \leq A\|x\|$ for all $x \in X$.

Theorem 2.10 Assume $A : H_1 \rightarrow H_2$ is a continuous, linear function and H_1, H_2 are Hilbert spaces. Then there is exactly one continuous, linear function $A^* : H_2 \rightarrow H_1$ such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x \in H_1$, $y \in H_2$.

Definition 2.20 1. The operator A^* constructed in theorem 2.10 is called the *adjoint* operator.

2. $A : H \to H$ is called a *hermitian* operator if $A = A^*$, i. e. $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x, y.
3. If A is a hermitian operator, then we have $\langle Ax, x\rangle \in \mathbf{R}$. A is a *positive* operator if $\langle Ax, x\rangle > 0$ for all $x \in H$.

2.1.5 Exercises

Exercise 2.1 Define $\mathbf{H}$ as the set of the complex 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

This defines the quaternion algebra $\mathbf{H}$. Show that:

1. $\mathbf{H}$ is a two-dimensional complex vector space with basis
$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
2. $\mathbf{H}$ is a four-dimensional real vector space with basis:
$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Quaternions are further investigated in the exercises of the next chapter.

Exercise 2.2 Show that all finite subsets of a Hausdorff space are closed.

Exercise 2.3 Define a neighborhood basis as follows: A system of neighborhoods $\mathcal{W} = \{W\}$ is called a *neighborhood basis* if every neighborhood $U(x)$ contains a $W \in \mathcal{W} : W \subset U(x)$.

Prove the following statement:

Given a set $\mathcal{W}$ of subsets of $V \subset X$ that satisfies the following conditions:

- $\emptyset \in \mathcal{W}$
- The intersection of any finite number of sets from $\mathcal{W}$ is the union of sets from $\mathcal{W}$.
- The union of all sets $V \in \mathcal{W}$ is X.

then $\mathcal{W}$ is the neighborhood basis of the topology $\mathcal{V}$ obtained by taking all unions of elements in $\mathcal{W}$.

Exercise 2.4 Assume that X is a normed space. Show that the set of all open balls $U(x, r) = \{y \in X | \|x - y\| < r\}$ where $x \in X$ and $r > 0$ form a neighborhood basis.

Exercise 2.5 [Pythagoras] Assume x and y are orthogonal elements of a Hilbert space H. Show that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

Exercise 2.6 Assume H is a complex Hilbert space and $A : H \to H$ is a linear map. Show that A is a hermitian operator if and only if $\langle Ax, x\rangle$ is real for all $x \in H$. (Hint: use that if $\langle Ax, x\rangle = 0$ for all $x \in H$ then $A = 0$.)

Exercise 2.7 Use the Cauchy-Schwarz inequality to show that if A is hermitian and $A \geq 0$ then we have for all $x, y \in H$:

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle \langle Ay, y\rangle .$$

Exercise 2.8 [Riesz Representation Theorem] Let H be a Hilbert space. For each linear continuous functional $f : H \to \mathbf{C}$ there is a unique element $y_f \in H$ such that: $f(x) = \langle x, y_f\rangle$ for all $x \in H$. (Hint: Define $M = \{x \in H | f(x) = 0\}$ and show that $M^\perp = H/M$ is one-dimensional.)

Exercise 2.9 Assume $\langle\ ,\ \rangle$ is the standard scalar product in a real finite dimensional vector space V. Assume further that $f : V \to V$ is a mapping such that $f(x) = Rx$ for a matrix R and $\langle x, x\rangle = \langle f(x), f(x)\rangle$ for all $x \in \mathbf{R}$. Then: $\langle x, y\rangle = \langle f(x), f(y)\rangle$.

A linear mapping that preserves the norm preserves also the angles. (Hint: Consider sums and differences of vectors).

2.2 Algebraic Theory of Groups

In this section we introduce the algebraic concept of a group and we will study some of the basic, algebraic properties of groups. For a more complete treatment the reader is referred to any standard text on algebra, for example [25].

2.2.1 Basic Concepts

Definition 2.21 A non-empty set G is called a *group* if there is a product

$$\circ : G \times G \to G; (g_1, g_2) \mapsto g_1 \circ g_2$$

with the following properties:

- $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$. We say that the product is *associative.*
- There is a unique element $e \in G$ such that $e \circ g = g \circ e = g$ for all $g \in G$. The element e is called the *identity element.*
- For all elements $g \in G$ there is a unique element g^{-1} such that $g^{-1} \circ g = g \circ g^{-1} = e$. We call g^{-1} the *inverse* of g.

Note that a group consists of two equally important components, the set G and the composition rule $\circ$. Normally we speak only of the group G, implicitly assuming that it is clear which composition rule is used. We use the notation $(G, \circ)$ only if we want to emphasize a certain composition rule.

Definition 2.22 If the product satisfies the condition

$$g_1 \circ g_2 = g_2 \circ g_1$$

for all $g_1, g_2 \in G$ then we say that G is a *commutative group* or an *abelian group.*

In the general case we often write $g_1 g_2$ instead of $g_1 \circ g_2$ and we denote the identity by 1. In the commutative case we write $g_1 + g_2$ instead of $g_1 \circ g_2$. In this case we denote the identity element by 0 and the inverse element of g by $-g$.

Definition 2.23 1. If G has a finite number of elements then we say that G is a *finite group;* otherwise we say that G is *infinite.*

2. The number of elements in the group is called the *order* of the group and it is denoted by $|G|$.

The simplest example of a group is a set with a single element e and the product $e \circ e = e$. The group with two elements e and a is obviously given by defining the product as: $e \circ e = e; e \circ a = a \circ e = a$ and $a \circ a = e$. Often it is convenient to describe the product rule with the help of a table. In the previous case this table is given by:

	e	a
e	e	a
a	a	e

Given a set G of three elements e, a and b we can make G into a group by defining the group multiplication $\circ$ with the following table:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(2.4)

Up to now we used G as a mere set of symbols. In our applications these symbols will always represent objects. One way to give the symbols in the last group a meaning is to consider the following mapping:

$$e \mapsto 0; a \mapsto 1; b \mapsto 2.$$

The multiplication table 2.4 now becomes:

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(2.5)

The group operation is addition modulo 3 in this interpretation of the group.

Instead of interpreting the group elements as numbers we could also give them a geometrical meaning. For this purpose we define the following map:

$$e \mapsto E; a \mapsto R; b \mapsto R^2$$

where E is the 2×2 identity matrix, R is the matrix

$$R = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

of 120 degree rotation and R^2 is the matrix of 240 degree rotation. In this case the group operation becomes the concatenation of two rotations or, equivalently the product of two matrices.

A slightly different geometric interpretation of G is the following: Consider set X consisting of the three points x, y and z with coordinates: $x = (0, 1/\sqrt{3}); y = (-1/2, 1/(2\sqrt{3}))$ and $z = (1/2, 1/(2\sqrt{3}))$. As group elements we use the three functions $e, a, b : X \to X$ defined as:

$$e(p) = p \text{ for all } p \in X$$

$$\begin{array}{lll} a(x) = y; & a(y) = z; & a(z) = x \\ b(x) = z; & b(y) = x; & b(z) = y \end{array}$$

Geometrically e, a and b act again as rotations but we can also interpret them as permutations acting on the set X. The situation where X is an ordinary set and G is a group of functions $f : X \to X$ will be treated extensively in the following.

The following examples introduce groups that will be studied in detail in the following chapters.

Examples 2.2 1. **The additive group of real numbers:** The set of real numbers becomes a group under the usual addition: $x_1 \circ x_2 = x_1 + x_2$. The identity is the number zero and the inverse of x is $-x$. This group will be denoted by $\mathbf{R}$.

2. **The additive group of integers:** The set of integers under the usual addition is a group. It is denoted by $\mathbf{Z}$.

3. **The multiplicative group of non-zero real numbers:** The set of the real numbers without the number zero is a group under the usual multiplication. The identity is the number one and the inverse element of x is $x^{-1} = 1/x$. This group is denoted by $\mathbf{R}_0$.

4. **The multiplicative group of positive real numbers:** The set of the positive real numbers is a group under the usual multiplication. This group is denoted by $\mathbf{R}_+$.

5. **The Symmetrical Group:** Assume X is a set consisting of N elements. A bijective map $p : X \to X$ is called a *permutation* of X. The permutations of X form a group under the usual composition of functions. This group is called the *symmetrical group* and it is denoted by S_N.

6. **The n-dimensional space:** The set of all $n-$dimensional vectors with real components is a group under the normal addition of vectors. This group will be denoted by $\mathbf{R}^n$. Note that also the set of vectors (without the group operation) is denoted by $\mathbf{R}^n$.

7. **Rotations in 2-D:** The rotation R_ϕ is defined as the mapping $R_\phi : \mathbf{R}^2 \to \mathbf{R}^2; (x, y) \mapsto (x', y')$ where x' and y' are defined as:

$$x' = x \cos\phi - y \sin\phi \quad (2.6)$$

$$y' = x \sin\phi + y \cos\phi \quad (2.7)$$

The product $R_{\phi_2} \circ R_{\phi_1}$ is the rotation obtained by first performing the rotation R_{ϕ_1} and then the rotation R_{ϕ_2}. We find that $R_{\phi_1+\phi_2} = R_{\phi_1} \circ R_{\phi_2}$. It is easy to see that these rotations define a group under the usual composition of mappings. This group is denoted by $SO(2)$.

8. **Complex Numbers:** The complex numbers under addition and the non-zero complex numbers under multiplication form groups. These groups are denoted by **C** and $\mathbf{C}_0$ respectively. The unit circle, i.e. the set $C = \{z : |z| = 1\}$ forms a group under complex multiplication.
9. **The open interval** $[0, a)$: The open interval is a group under addition modulo a.
10. **The general linear group:** The set of complex matrices of size $n \times n$ with non-zero determinant is a group under ordinary matrix multiplication. It is denoted by $GL(n, \mathbf{C})$. In the same way we define $GL(n, \mathbf{R})$ as the set of real $n \times n$ matrices with non-zero determinant.
11. **The special linear groups:** The sets of real or complex matrices of size $n \times n$ with determinant 1 are groups. These groups are denoted by $SL(n, \mathbf{R})$ and $SL(n, \mathbf{C})$ respectively.
12. **The orthogonal groups:** An *orthogonal* matrix R is an element of $GL(n, \mathbf{R})$ that satisfies the condition: $R'R = E_n$ where R' is the transpose of R and E_n is the $n \times n$ identity matrix. The set of all orthogonal matrices defines a group under matrix multiplication. It is denoted by $O(n)$.
13. **The special orthogonal groups:** The set of all orthogonal matrices $R \in O(n)$ with $\det R = 1$ is a group, the special orthogonal group $SO(n)$. Note that we denoted both, the matrix group and the group of 2-D rotations by $SO(2)$. This reflects the known property that rotations can be described by matrices from $SO(n)$ after the introduction of an appropriate coordinate system. We could also view these two groups as realizations of the same abstract group.
14. **The unitary groups:** Recall that a complex matrix U is *unitary* if $U^*U = E_n$ where U^* is the transposed, conjugate complex of U. The set of all $n \times n$ unitary matrices forms a group under matrix multiplication. This group is denoted by $U(n)$.
15. **The special unitary groups:** The set of all unitary matrices of size $n \times n$ with $\det U = 1$ is also a group, the special unitary group. These groups are denoted by $SU(n)$.
16. **Euclidian motion groups:** Recall that a euclidian motion is a rotation followed by a translation. The euclidian motions in n-dimensional space form a group that is denoted by $M(n)$. After introducing coordinates we can describe a euclidian motion as $y = Rx + T$ where x and y are the coordinate vectors of the original point and its image. Here $R \in SO(n)$ is a rotation matrix and T is a translation vector. If we introduce the $(n+1)$–dimensional vector $\tilde{x}$ as $\tilde{x} = \begin{pmatrix} x & 1 \end{pmatrix}$ then we can also write $\tilde{y} = M\tilde{x}$ where M is the matrix: $M = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$.

2.2.2 Subgroups and Cosets

Definition 2.24 1. A non-empty subset H of a group G is called a *subgroup* if $g_1 g_2^{-1} \in H$ for all g_1 and g_2 in H.

2. If $H \neq G$ then we say that H is a *proper* subgroup of G.

We find $e = g_1 g_1^{-1} \in H$ and therefore also $g^{-1} = eg^{-1} \in H$. Thus H is itself a group under the composition rule inherited from the larger group G. If H is a subgroup of G then we write $H < G$.

Examples 2.3

1. The subset $H = \{e\}$ is the smallest subgroup of a group. G is its largest subgroup. G and $\{e\}$ are the *trivial* subgroups of G.
2. The intersection of two subgroups is a subgroup.
3. Assume a is a fixed element of G and define $H = \{a^n | n \in \mathbf{Z}\}$. Then H is a subgroup of G. Such subgroups are called *cyclic*. The element a is called the *generator* of H.
4. The set of 3-D rotations around the z-axis forms a subgroup of $SO(3)$.

We have the following relations between the groups introduced so far:

Theorem 2.11

1. $\mathbf{Z} < \mathbf{R}$
2. $\mathbf{R}_+ < \mathbf{R}_0$
3. $C < \mathbf{C}_0$
4. $SU(n) < U(n) < GL(n, \mathbf{C})$
5. $SO(n) < O(n) < GL(n, \mathbf{R})$
6. $SO(n) < SU(n)$
7. $O(n) < U(n)$
8. $GL(n, \mathbf{R}) < GL(n, \mathbf{C})$

Definition 2.25

1. Assume H is a subset of G and g is an arbitrary element of G, then we denote by Hg the set $\{hg | h \in H\}$. Such a set is called a *right coset* of H. *Left cosets* are defined in like manner.
2. An element of a coset is called a *representative* of the coset and by $\tilde{g}$ we denote the coset which contains g.

Theorem 2.12 The group G falls into pairwise disjoint cosets of H if $H < G$.

It is clear that G is the union of all its cosets and it only remains to show that different cosets are disjoint. To see this assume that g is a common member of the cosets Hg_1 and Hg_2. Then we find that $g = h_1 g_1 = h_2 g_2$ for some elements $h_1, h_2 \in H$. If hg_1 is an arbitrary element in Hg_1 then we find $hg_1 = hh_1^{-1}h_1 g_1 = hh_1^{-1}h_2 g_2 = h'g_2$ with $h' \in H$ and therefore we find that hg_1 is an element of Hg_2 which shows that the two cosets are equal.

Definition 2.26 Assume K is a subset of the group G. Then we say that K generates G if all elements $g \in G$ have the form: $g = a_1 ... a_n$ where $a_i \in K$ or $a_i^{-1} \in K$. In general the index n is different for different elements $g \in G$.

Thus a subset K generates G if all elements in G are products of elements in K or their inverses.

Examples 2.4 The group of permutations of three elements S_3 has six elements:

$$e=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} a=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} b=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$c=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} d=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} f=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The group operation is summarized in the following table:

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	f	c	d
b	b	e	a	d	f	c
c	c	d	f	e	a	b
d	d	f	c	b	e	a
f	f	c	d	a	b	e

(2.8)

The group is not commutative since $ac \neq ca$. The set $H = \{e, a, b\}$ is a cyclic subgroup and the subset $K = \{a, c\}$ generates S_3.

2.2.3 Normal Subgroup and Center

Definition 2.27 1. Let H be a subgroup of the group G. Consider each right coset Hg as a single element in the space of right cosets. The new space of cosets is called the *factor space* or the *space of cosets* of the subgroup H in G. It is denoted by G/H.

2. If the number of elements in G/H is finite then we call this number the *index of* H *in* G and denote it by $|G/H|$.

For finite groups and their subgroups we have the following relations:

Theorem 2.13 Let G be a finite group and $H < G$ be a subgroup. Then we have:

1. $|G| = |H|\,|G/H|$.
2. The order of H divides the order of G.
3. G has only trivial subgroups if the order of G is a prime number.

Examples 2.5 1. Take $G = \mathbf{R}$ and $H = \mathbf{Z}$. A representative $\tilde{g} \in G/H$ has the form: $\tilde{g} = g_\alpha = \{\alpha + n | n \in \mathbf{Z}, 0 \leq \alpha < 1\}$.

2. If $G = \mathbf{Z}$ and H is the set of all even integers then $|G/H| = 2$ and we can think of the elements of G/H as the sets of even and odd integers.

Normally left cosets and right cosets are different. The subgroups for which left cosets and right cosets are identical are called normal:

Definition 2.28
1. Let G be a group and $H < G$. H is called a *normal divisor* or a *normal subgroup* of G if $gH = Hg$ for all $g \in G$.
2. $\{e\}$ and G are called the *trivial normal divisors* of G. All other normal subgroups are called *nontrivial.*
3. A group is called *simple* if it has only trivial normal subgroups.

Theorem 2.14 Assume H is a normal subgroup of G. On the factor space G/H we define the multiplication: $g_1Hg_2H = (g_1g_2)H$. This definition is independent of the elements g_1 and g_2 and G/H becomes a group under this operation.

Definition 2.29 The set G/H together with the multiplication defined in theorem 2.14 is called the *factor group* G/H.

Examples 2.6
1. $G = GL(n, \mathbf{C}), H = SL(n, \mathbf{C})$. H is a normal subgroup of G. The map $\lambda : G/H \rightarrow \mathbf{C}_0; \tilde{g} \mapsto \det g$ is bijective and satisfies $\lambda(\tilde{g_1}\tilde{g_2}) = \lambda(\tilde{g_1})\lambda(\tilde{g_2})$ for all $\tilde{g_1}, \tilde{g_2} \in G/H$. An element in G/H has the form $\tilde{g}_\alpha = \{g \in GL(n, \mathbf{C}) | \det g = \alpha\}$.
2. $G = \mathbf{R}, H = \mathbf{Z}, G/H = \{g_\alpha | \alpha \in [0, 1)\}$ with $g_\alpha = \{\alpha + n | n \in \mathbf{Z}\}$. The addition in G/H becomes: $g_{\alpha_1} + g_{\alpha_2} = g_{[\alpha_1+\alpha_2]}$ where we defined: $[\alpha_1 + \alpha_2] = \alpha_1 + \alpha_2 \mod 1$.

Finally we introduce a subgroup that describes "how commutative" a group is.

Definition 2.30 The *center* of a group is the set of all group elements that commute with all group elements, it will be denoted by $Z(G)$.
$Z(G) = \{g_0 \in G | g_0 g = g g_0 \text{ for all } g \in G\}$.

Theorem 2.15 The center $Z(G)$ of a group has the following properties:

1. $Z(G)$ is a group.
2. $Z(G)$ is commutative.
3. $Z(G)$ is a normal subgroup of G.

2.2.4 Homomorphisms and Isomorphisms

In this section we will discuss mappings between groups which preserve the group operation.

Definition 2.31
1. A mapping $f : G \rightarrow G'$ of a group G into a group G' is called a *homomorphism* if it preserves the group operation, i. e.: $f(g_1g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.
2. A bijective homomorphism is called an *isomorphism.*

3. Two groups G and G' are called *isomorphic* if there is an isomorphism $f : G \to G'$. In this case we write $G \cong G'$.

Two isomorphic groups are thus identical as far as their algebraic properties are concerned. Examples are the group of 2-D rotations, the group of all complex numbers of magnitude one and the group of all real, orthogonal 2×2 matrices.

Given two groups G and G' and a homomorphism f between them it is easy to show that the set of all elements in G that are mapped to the identity element in G' forms a subgroup of G. We define:

Definition 2.32 $\ker(f) = \{g \in G | f(g) = e'\}$ is called the *kernel* of f. e' is in this case the identity in G'.

Homomorphisms have the following properties:

Theorem 2.16 A homomorphism f has the following properties:

1. $f(e) = e'$
2. $f(g^{-1}) = f(g)^{-1}$
3. $\ker(f)$ is a normal subgroup of G.
4. f is an isomorphism if f is surjective and if $\ker(f) = \{e\}$.

The case where an isomorphism maps a group onto itself is so important that we consider this case separately:

Definition 2.33 1. An isomorphism of G onto itself is called an *automorphism*.

2. The automorphisms form a group under the usual composition rule $(f_2 f_1)(g) = f_2(f_1(g))$, this group is called the *automorphism group* of G and it is denoted by $A(G)$.
3. An automorphism of the form: $f(g) = g_0^{-1} g g_0$ for a fixed element $g_0 \in G$ is called an *inner automorphism*.
4. The inner automorphisms form a subgroup of $A(G)$, the *inner automorphism group* of G. It is denoted by $A_i(G)$.

An important example of a homomorphism is the projection from the original group onto a factor group. We define:

Definition 2.34 Assume that G is a group and H is a normal subgroup of G and G/H is the factor group. The mapping: $p : G \to G/H; g \mapsto Hg$ is called the *canonical homomorphism* or the *natural mapping*.

We saw that the kernel of a homomorphism is a normal subgroup and we can therefore construct the factor group $G/\ker f$. For this factor group we find the following important theorem which decomposes an arbitrary homomorphism into the projection and an isomorphism.

Theorem 2.17 Assume $f : G \to G'$ is a surjective homomorphism and $H = \ker(f)$. Then

1. $G' \cong G/H$
2. $f = gp$ where g is an isomorphism and p is the canonical homomorphism (see Figure 2.1).

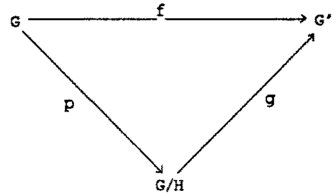

Figure 2.1: Factorization of a homomorphism

2.2.5 Transformation Groups

In this section we will generalize the example from page 17 where we considered permutations and rotations that left a triangle invariant. We considered the set of the three corner points of a triangle and a set of rotations that mapped a given corner point into another corner point. The same concept underlies the definition of the symmetrical group (see page 17 where we had a set of N elements and the group elements of S_N where the bijective mappings of this set. We generalize these examples in the concept of a transformation group.

Definition 2.35 Let X be any set.

1. A bijective map of X is called a *transformation* of X.
2. If g is a transformation then we write xg or x^g instead of $g(x)$ and we say that g is a *right transformation.* If we write gx or ${}^gx$ instead of $g(x)$ then we say that g is a *left transformation.*
3. Assume g_1 andg_2 are right transformations, then we define the *product of* g_1 *and* g_2 as the transformation which is obtained by applying first g_1 and then g_2. We write $x(g_1g_2) = (xg_1)g_2$.
4. The set of all transformations forms a group under the product defined above. This group is denoted by $G(X)$.
5. Any subgroup of $G(X)$ is called a *transformation group of the set* X.
6. The pair (X, G) with a set X and a transformation group G is called a *space* X *with transformation group* G.
7. The subgroup of all linear transformations of X is denoted by $GL(X)$.

Examples 2.7 1. The symmetric group S_N is the special transformation group that belongs to a finite set with N elements.

2. Assume $\mathcal{C}_r$ is a circle with radius r around the origin and $SO(2)$ is the group of 2-D rotations. The pair $(\mathcal{C}_r, SO(2))$ is a space with a transformation group. It is also easy to see that the unit sphere and the group $SO(n)$ form a transformation group.

3. Let $\mathbf{R}$ be the real axis and S the group of transformations defined by $sx = ax + b$ with $a \neq 0$. S is called the *group of linear transformations of the line.* The subgroup S^1 of all transformations of the form $sx = x + b$ is called the *group of translations of the line* and the subgroup S^2 defined by $sx = ax$ is the *group of dilatations of the line.* $(\mathbf{R}, S), (\mathbf{R}, S^1)$ and $(\mathbf{R}, S^2)$ are transformation groups.
4. Let Π^1 be the complex plane together with an additional point ∞. Let F^1 be the set of transformations $f : z \mapsto z' = f(z)$ defined as:

$$z' = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty \text{ and } z \neq -d/c \\ \infty & \text{if } z = -d/c \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

where a, b, c, d are arbitrary complex numbers with $ad - bc \neq 0$. Then (Π^1, F^1) form a space with a transformation group and F^1 is called the *group of fractional-linear transformations of the plane* Π^1.

Two points in the space belonging to a transformation group can be regarded as similar if there is a transformation that maps one point onto the other. For example all points on a circle with a certain radius are similar with respect to the rotation group. We formalize this in the following definition:

Definition 2.36 Assume (X, G) is a space X with transformation group G, assume further that x is an element in X. Then we call the set $\{xg | g \in G\}$ an *orbit* or a *trajectory* relative to G.

It is easy to see that "being in the same orbit" defines an equivalence relation and X is therefore the disjoint union of orbits.

Definition 2.37 The space X is called *transitive* or *homogeneous* if X consists of one orbit.

Homogeneous spaces are important in our applications since we can identify points in the underlying space X with equivalence classes of group elements in the following way:

Assume x_0 is an arbitrary but fixed point in the space X and define the subset $H \subset G$ as the set of all elements that leave x_0 fixed:

$$H = \{g \in G : x_0 g = x_0\}$$

For an element $g \in G$ we define the subset H^g as $g^{-1}Hg$ and K as $K = \{H^g : g \in G\}$. On G we define an equivalence relation by $g_1 \equiv g_2$ if they are elements of the same set H^g. This means that there is a $g \in G$ and elements $h_i \in H$ such that $g_i = g^{-1}h_i g (i = 1, 2)$. This is obviously an equivalence relation in G and G is thus the disjunct union of the different elements in K.

Now assume that $y \in X$. Then we can find an element $g_y \in G$ such that $y = x_0 g_y$ since X is homogeneous. We define the mapping i as:

$$i : X \to K; y \mapsto H^{g_y}$$

This map is well defined since $y = x_0 g = x_0 g'$ implies $g'g^{-1} \in H$ and therefore $H^g = H^{g'}$ and it is also bijective. Another way to map X into the group G is to identify a point $y \in$

X with all transformations $g \in G$ that map x_0 into y. Obviously these transformations are the elements of the right coset Hg_y where g_y is defined as above. Consequently we can identify the space X (on which G operates) with the factor space of all right cosets of H.

In one of the simplest examples X is the unit circle and $G = SO(2)$. In this case we identify X with $SO(2)$ by mapping the complex number $e^{i\phi}$ to the rotation with rotation angle ϕ. If X is the unit sphere in 3-D space and if we select x_0 as the north pole then we find that H is the set of all 3-D rotations that leave the north pole fixed. The elements in H are thus represented by matrices of the form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$ where $R \in SO(2)$ is a 2-D rotation.

One important example of a transformation group can be obtained by using as X the group G itself. This example is formulated in the next definition and theorem (see also definition 2.33)

Definition 2.38 1. Two elements g_1 and g_2 in G are called *conjugate* if there is an element $g \in G$ such that $g_2 = g^{-1}g_1g$

2. The set of all elements in G which are conjugate to one fixed element $g \in G$ is called a *conjugacy class.*

From the definition of an inner automorphism and the definition of a conjugacy class we obtain immediately the following theorem:

Theorem 2.18 1. The orbit of an element g relative to the inner automorphism group $A_i(G)$ is the conjugacy class of g.

2. Define the homomorphism $f : G \to A_i(G); g \mapsto a_g$ where a_g is the inner automorphism $a_g(h) = g^{-1}hg$. Then we find that $\ker(f) = Z(G)$.

3. A subgroup $H < G$ is a normal subgroup if and only if H is invariant under all inner automorphisms $a_g, g \in G$.

2.2.6 Direct Product of Groups

The last concept from the elementary, algebraic theory of groups is the direct product of a finite number of groups which will be introduced in this section.

Definition 2.39 Assume $G_1, ..., G_n$ is a set of groups. Then $G_1 \times ... \times G_n$ is the set of all sequences $\{(g_1, ..., g_n)|g_i \in G_i\}$. We make this set product into group by defining a composition rule $\circ$ as $(g_1, ..., g_n)\circ(g_1', ..., g_n') = (g_1g_1', ..., g_ng_n')$ where g_ig_i' is the group operation defined on the group G_i. We will also write $(g_1, ..., g_n)(g_1', ..., g_n')$ instead of $(g_1, ..., g_n) \circ (g_1', ..., g_n')$. The cartesian product of the groups G_i together with this composition rule is called the *direct product of the groups.* The identity element is given by $(e_1, ..., e_n)$ where e_i is the identity element in G_i.

Theorem 2.19 If we define the mappings $f_i : G_i \to G_1 \times ... \times G_n; g_i \mapsto (e_1, ..., g_i, ..., e_n)$ then we can identify G_i with a subgroup of the product $G = G_1 \times ... \times G_n$ and we write g_i instead of $(e_1, ..., g_i, ..., e_n)$. With these notations it is easy to show that all elements $g \in G$ are of the form: $g = g_1...g_n$ and that any two elements $g_j, g_k \in G$ with $j \neq k$ satisfy $g_jg_k = g_kg_j$.

Definition 2.40 Assume $G_1, ..., G_n$ are subgroups of G which satisfy the following conditions: All elements $g \in G$ are of the form $g = g_1...g_n$ with $g_i \in G_i$ and they also satisfy the condition $g_j g_k = g_k g_j$ if $j \neq k$. Then we say that G is the *direct product* of the subgroups $G_1...G_n$.

2.2.7 Exercises

Exercise 2.10 Show the group properties for the following groups:

1. **A finite group:** The set $G_0 = \{1, i, -1, -i\}$ is a finite group under the usual complex multiplication.
2. **Multiples of a fixed natural number:** The set is given by $\{m = np | n \in \mathbf{Z}\}$, the composition is the addition of natural numbers. This group is denoted by $\mathbf{Z}_p$.
3. **The p^{th} roots of unity:** This group consists of the numbers $e^{2\pi ik/p}$ and the composition rule is complex multiplication (k is an integer and p is a natural number). This group is denoted by Ω_p.
4. **A dynamical system:** Assume the function $f(t)$ describes the state of a system at time t, assume further that t_0 is a fixed constant and define the operator T as $(Tf)(t) = f(t + t_0)$. Show that the operators $\{T^n | n \in \mathbf{Z}\}$ form a group under the composition rule $T^n T^m = T^{n+m}$.

Exercise 2.11 1. Show that $SL(n, \mathbf{C})$ is a normal subgroup of $GL(n, \mathbf{C})$.

2. Show that $GL(n, \mathbf{C})/SL(n, \mathbf{C})$ and $\mathbf{C}_0$ (i.e. the multiplicative group of non-zero complex numbers) are isomorphic. What is the isomorphism?
3. Are $\mathbf{R}_0$ and $GL(n, \mathbf{R})/SL(n, \mathbf{R})$ isomorphic?

Exercise 2.12 What is $SO(3)/SO(2)$?
Hint: A 3-D rotation can be described by its rotation axis and its rotation angle.

Exercise 2.13 Prove theorem 2.15.

Exercise 2.14 Show that the center $Z(GL(n, \mathbf{C})$ is given by the matrices λE_n where $\lambda \in \mathbf{C}$ and E_n is the unit matrix in G. (Hint: Find all matrices that commute with diagonal matrices that have n different non-zero entries in the diagonal.) What is $Z(GL(n, \mathbf{R}))$?

Exercise 2.15 Prove the properties of a homomorphism mentioned in theorem 2.16.

Exercise 2.16 Show that $f : GL(n, \mathbf{C}) \rightarrow \mathbf{C}; g \mapsto \det g$ is a homomorphism.

Exercise 2.17 Show that every group G is isomorphic to the group of all right translations on G.

Exercise 2.18 $[0, 1)$ together with the addition modulo 1 is isomorphic to the unit circle $\left\{e^{2\pi i\phi}\right\}$ with complex multiplication.

Exercise 2.19 $[0,1)$ together with addition modulo 1 is isomorphic to $SO(2,\mathbf{R})$.

Exercise 2.20 Assume G is a finite group with p elements, $|G| = p$ with a prime number p. What are the cyclic subgroups of G?

Exercise 2.21 Define $\mathbf{H}$ as the set of the complex 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

1. Show that this set is a group with matrix addition.
2. Show that the non-zero elements in $\mathbf{H}$ form a group under matrix multiplication. Is this group commutative?
3. Show that the matrices
$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$
with $a \in \mathbf{R}$ form a subgroup (under addition and multiplication).
4. Show that $\mathbf{R}$ is the center of $\mathbf{H}$ under multiplication.
5. Show that the matrices
$$\begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}$$
with $c \in \mathbf{C}$ form a subgroup (under addition and multiplication).
6. Show that
$$\mathbf{C}^2 \to H; (a,b) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$
is an isomorphism.

The previous sequence of exercises shows that we have the following sequence of extensions of $\mathbf{R} : \mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$. The set $\mathbf{H}$ together with addition and multiplication is called the *quaternion algebra* $\mathbf{H}$. The theorem of Frobenius shows that $\mathbf{C}$ and $\mathbf{H}$ are essentially the only extensions of $\mathbf{R}$. For a proof see [40]. More information on quaternions, their relation to other number systems and a historical background can be found in [18].

2.3 Topological Groups

In this section we combine the algebraic concept of a group and the topological concept of continuity to define topological groups, i.e. groups in which the group operations are continuous maps. We will also list some of the basic properties of these groups and we will study some matrix groups that will play a central role in the following chapters.

2.3.1 Topological Groups

We first introduce some notation: Assume $U \subset G$ is a subset of G. Then we define the following subsets:

$$U^{-1} = \left\{g^{-1} | g \in U\right\},$$
$$g_0 U = \{g_0 g | g \in U\},$$

and if U_1 and U_2 are subsets of G then we define $U_1 U_2$ as the subset

$$U_1 U_2 = \{gh | g \in U_1, h \in U_2\}.$$

We now define a topological group as a group with continuous group operations:

Definition 2.41 A group G is called a *topological group* if:

- G is a Hausdorff space and if
- the group operation $\circ : G \times G \to G; (g,h) \mapsto g \circ h$ and the inversion $G \to G; g \mapsto g^{-1}$ are continuous maps.

Theorem 2.20
1. The mapping $f_i : g \mapsto f_i(g) = g^{-1}$ is a homeomorphism.
2. The left translations $g \mapsto g_0 g$ and the right translations $g \mapsto g g_0$ are homeomorphisms.

To see the first part note that f_i is continuous according to the definition and that $f_i^{-1} = f_i$. In order to show the second part we use the fact that $g \mapsto g g_0^{-1}$ is the inverse of the translation $g \mapsto g g_0$.

Examples 2.8
1. An ordinary group equipped with the discrete topology is a topological group.
2. The real and complex spaces $\mathbf{R}^n$ and $\mathbf{C}^n$ are additive, topological groups under their natural topologies.
3. The non-zero real and the complex numbers $\mathbf{R}_0$ and $\mathbf{C}_0$ are multiplicative, topological groups under their natural topologies.
4. Consider $GL(n, \mathbf{R})$ and $GL(n, \mathbf{C})$ as subspaces of $\mathbf{R}^{n^2}$ and $\mathbf{C}^{n^2}$ respectively. As subsets they inherit the topology from the larger spaces. This topology is called the **natural topology** of these groups. These groups form topological groups since polynomials are continuous functions.

We continue by defining topologies on subgroups and factor groups of topological groups:

Definition 2.42 Assume H is a subgroup of a topological group G. Then we make H into a topological group by providing H with the topology of a subspace. An open set V in H is thus a subset of the form $V = U \bigcap H$ where U is an open subset of G.

The factor group is given a topology by the following construction:

Definition 2.43 Assume G is a topological group and H is a subgroup of G. G/H is the set of right cosets and $p : G \to G/H; g \mapsto Hg$ is the natural or canonical mapping. Then we define a topology on G/H as the set of all images of open sets $U \subset G$ under p. A subset $V \subset G/H$ is therefore open if there is an open set $U \subset G$ with $V = p(U)$.

Theorem 2.21
1. We say that a map is open if the image of every open set is open. The natural mapping p is thus open and continuous.
2. If H is a closed subgroup of G then G/H is separated.
3. If H is a closed, normal subgroup of G then G/H is a topological group.

Next we have to define the mappings that are compatible with the topological and algebraical properties of groups:

Definition 2.44 Assume G and G' are topological groups and $f \to G'$ is a mapping. Then we say that

1. f is a *continuous homomorphism* if f is a homomorphism and if it is continuous.
2. f is an *continuous isomorphism* if f is an isomorphism and if it is continuous.
3. f is a *topological isomorphism* if f is an isomorphism and if it is a homeomorphism.

1. Two groups are called *topological isomorphic* if there is a topological isomorphism between them.
2. A topological isomorphism $f : G \to G$ is called a *topological automorphism.*

The theorem 2.17 about the factorization of a homomorphism now becomes:

Theorem 2.22
1. If $f : G \to G'$ is a continuous surjective homomorphism and $H = \ker(f)$, then:
   - H is a closed, normal subgroup of G.
   - $f = gp$ where g is a continuous isomorphism of G/H onto G'.
2. If f is also open then g is a topological isomorphism.

In the last definition of this section we define continuous vector-valued functions and continuous linear operators on the group.

Definition 2.45
1. Assume $f : G \to \mathbf{C}^n; g \mapsto (f_1(g), \ldots, f_n(g))$ is a complex valued vector function then we say that f is a *continuous function* if all functions f_i are continuous.
2. Denote the space of all linear operators on a vector space X by $L(X)$. If X is finite dimensional then the elements of $L(X)$ are described by matrices. Now assume that $A : G \to L(X); g \mapsto A(g)$ is an operator valued function. For each $g \in G$ the function value $A(g)$ is a linear operator on a finite dimensional vector space X. Then we say that A is a *continuous operator* if the mapping $g \mapsto A(g)x$ is continuous for all $x \in X$.

It is easy to see that a linear operator is continuous if and only if its matrix elements $(a_{ij})(g)$ are continuous, complex-valued functions in any basis $e_1, \ldots, e_n$ of X.

2.3.2 Some Matrix Groups

In this section we derive some important properties of the matrix groups that will be most important in what follows.

In section 2.2 we introduced the following matrix groups: $GL(n,\mathbf{R})$, $GL(n,\mathbf{C})$, (the general linear groups), $SL(n,\mathbf{R})$, $SL(n,\mathbf{C})$, (the special linear groups), $O(n)$, $SO(n)$, (the orthogonal and special orthogonal groups) and $U(n)$ and $SU(n)$ (the unitary and special unitary groups). There we mentioned that these sets form groups under the usual matrix multiplication. From the definition of these groups it is also easy to see that we have the following subgroup relations among these groups (see also theorem 2.11):

1. $SO(n) < O(n)$,
2. $SO(n) < SL(n,\mathbf{R}) < GL(n,\mathbf{R})$,
3. $SU(n) < U(n)$ and
4. $SU(n) < SL(n,\mathbf{C}) < GL(n,\mathbf{C})$.

We also mentioned (see section 2.11) that $SL(n,\mathbf{R})$ and $SL(n,\mathbf{C})$ are normal subgroups of the general linear groups $GL(n,\mathbf{R})$ and $GL(n,\mathbf{C})$ respectively. Furthermore we saw that $\mathbf{C}_0$ and $\mathbf{R}_0$ (the non-zero complex and real numbers) were isomorphic to the factor groups $GL(n,\mathbf{C})/SL(n,\mathbf{C})$ and $GL(n,\mathbf{R})/SL(n,\mathbf{R})$ respectively. It can also be shown that the mapping

$$f : GL(n,\mathbf{C}) \to \mathbf{C}_0; g \mapsto \det g$$

is an open, continuous isomorphism. The groups $\mathbf{C}_0$ and $GL(n,\mathbf{C})/SL(n,\mathbf{C})$ are thus topologically isomorphic. In this section we will derive some important properties of these groups.

Theorem 2.23 $U(n), SU(n), O(n)$ and $SO(n)$ are compact groups.

We consider only the unitary case since the compactness of the other groups can be shown by similar arguments:

From elementary analysis it is known that a subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded. From the orthogonality relations $\sum_{l=1}^n u_{lk}\overline{u_{lj}} = \delta_{ij}$ we find that $U(n)$ is closed since the left side of the equation is a continuous function in the variables u_{kl} and the right side is a closed set. From the orthogonality we find also $\sum_{k=1}^n \sum_{l=1}^n |u_{lk}|^2 = n$ and the matrices in $U(n)$ are thus all contained in a sphere of radius n. $U(n)$ is thus closed and bounded and therefore compact. $SU(n) = U(n) \bigcap SL(n,\mathbf{C})$ is a subgroup of $U(n)$ and therefore bounded. $U(n)$ and $SL(n,\mathbf{C})$ are closed from which the compactness of $SU(n)$ follows.

The full unitary (orthogonal) group consists of two components and one such component is the special unitary (orthogonal) group.

Theorem 2.24 1. $U(n)$ and $U(1) \times SU(n)$ are homeomorphic.

2. $O(n)$ and $\{-1,1\} \times SO(n)$ are homeomorphic.

In the unitary case the mapping is given by $(\det U, U_1) \mapsto U$ where U_1 is obtained from U by dividing the first column of U by $\det U$. The determinant $\det U$ is in $U(1)$ since $1 = \det E_n = \det(U^*U) = \det U^* \det U = |\det U|^2$. In the real case we observe that $\det R \in \{-1, 1\}$ and the rest follows by a similar argument as above.

The group $SL(2, \mathbf{R})$ is locally compact but no longer compact. It contains, for example, all matrices of the form $\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$ with $t \in \mathbf{R}_0$. $SL(2, \mathbf{R})$ contains thus a group that is homeomorphic to $\mathbf{R}_0$. But from the Heine-Borel theorem it follows easily that this set is not compact since it is not bounded.

Let us now consider the case $n = 2$ in more detail. In this case we can calculate the form of the elements in $SU(2)$ and $SO(2)$.

Theorem 2.25 1. $SU(2)$ consists of all matrices of the form

$$U = \begin{pmatrix} \overline{b} & -\overline{a} \\ a & b \end{pmatrix}$$

with complex constants a and b that satisfy $\|a\|^2 + \|b\|^2 = 1$

2. $SO(2)$ consists of all matrices of the form

$$R = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

with real constants x, y and $x^2 + y^2 = 1$

This can be seen by a direct calculation. In the $SO(2)$ case the constants x and y are of course given by $x = \cos\phi$ and $y = \sin\phi$ for some angle ϕ.

From this theorem we find also that $SU(2)$ is topologically equivalent to the unit sphere in $\mathbf{R}^4$ and $SO(2)$ is topologically equivalent to the unit circle:

Theorem 2.26 1. Describe an element U in $SU(2)$ by the matrix

$$U = \begin{pmatrix} \overline{b} & -\overline{a} \\ a & b \end{pmatrix}$$

and set $a = x_1 + ix_2$ and $b = x_3 + ix_4$.
The mapping $f : SU(2) \to \mathbf{R}^4; U \mapsto (x_1,\ x_2,\ x_3,\ x_4)$ is a homeomorphism of $SU(2)$ onto the unit sphere in $\mathbf{R}^4$.

2. The mapping $f : SO(2) \to U(1); R \mapsto e^{i\phi}$ is an topological isomorphism. Observe that $U(1)$ is identical to the unit circle in $\mathbf{C}$.

Next we will try to describe $SO(3)$ in different ways. When we introduced $SO(n)$ we viewed it as a subset of $\mathbf{R}^{n^2}$ that was defined by the equations $R'R = E_n$. $SO(3)$ is therefore the set of all solutions of the nine equations $r_i'r_j = \delta_{ij}(1 \leq i, j \leq 3)$ where r_i is the i-th column of R and r_i' is the i-th row. This is however not a very useful characterization. In the next theorem we will therefore characterize it with the help of three parameters, the so-called Euler angles:

Theorem 2.27 1. For every rotation $R \in SO(3)$ there are three angles φ_1, φ_2 and θ such that $R = R(\varphi_2)R(\theta)R(\varphi_1)$ where $R(\varphi)$ is a matrix of the form:

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $R(\theta)$ has the form:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Furthermore: $0 \le \varphi_1, \varphi_2 \le 2\pi$ and $0 \le \theta < \pi$.

2. A rotation matrix R has the form:

$$R = \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \cos\theta\sin\varphi_1\sin\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\theta\cos\varphi_1\sin\varphi_2 \\ \sin\varphi_2\sin\theta \end{pmatrix.$$
$$\begin{matrix} -\cos\varphi_1\sin\varphi_2 - \cos\theta\sin\varphi_1\cos\varphi_2 & \sin\theta\sin\varphi_1 \\ \sin\varphi_1\sin\varphi_2 + \cos\theta\cos\varphi_1\cos\varphi_2 & -\sin\theta\cos\varphi_1 \\ \cos\varphi_2\sin\theta & \cos\theta \end{matrix}\right)$$

Assume that under the rotation R the axes x, y, z go to x', y', z'. By l we denote the intersection of the xy-plane with the $(x'y')$-plane. The angle between the x-axis and l is φ_1 and the angle between l and x' is φ_2, finally the angle between the z- and the z'-axis is θ. In the first step we rotate around the z-axis so that x goes to l. Then we rotate around l to move z to z'. Finally we rotate around z' so that x goes to x' and y to y'.

Theorem 2.28 1. A rotation $R \in SO(3)$ can be described by its axis and the rotation angle (The rotation axis is a space of vectors x with $Rx = x$).

2. Assume $1, e^{i\theta}, e^{-i\theta}$ are the eigenvalues of R. The rotation angle of R is then given by $\cos\theta = (\text{trace(R) - 1})/2$.

3. For the rotation axis $v = (v_1, v_2, v_3)$ we find the ratio:

$$v_1 : v_2 : v_3 = R_{23} - R_{32} : R_{31} - R_{13} : R_{12} - R_{21}$$

To see this let λ_i and $v_i (i = 1, 2, 3)$ be the eigenvalues and eigenvectors of $R : Rv_i = \lambda_i v_i$. Since R is orthogonal we find that all eigenvalues have magnitude one: $|\lambda_i| = 1$. For the eigenvalues we find $\lambda_1\lambda_2\lambda_3 = \det R = 1$. Since R is a real matrix we find that the complex eigenvalues are conjugate complex. If there are complex eigenvalues (for example λ_2 and λ_3) then we find that the real eigenvalue λ_1 has value one. If there are only real eigenvalues then we find that $\lambda_i \in \{-1, 1\}$. We can thus find at least one eigenvalue equal to one. For this eigenvalue and the corresponding eigenvector we find $v = Rv$. v describes the rotation axis of R. Now let $\lambda_1 = 1, \lambda_2 = e^{i\theta}, \lambda_3 = e^{-i\theta}$ be the eigenvalues with eigenvectors v_1, v_2, v_3. These three vectors form an orthonormal set in $\mathbf{R}^3$ and we can go over to the new orthonormal system u_1, u_2, u_3 defined as

$$u_1 = v_1$$

$$u_2 = (v_1 + v_2)/\sqrt{(2)}$$
$$u_3 = i(v_1 - v_2)/\sqrt{(2)}$$

In this new coordinate system we find:

$$Ru_1 = u_1$$

$$Ru_2 = \cos\theta u_2 + \sin\theta u_3$$
$$Ru_3 = -\sin\theta u_2 + \cos\theta u_3.$$

The vector u_1 describes then the rotation axis and θ the rotation angle. The formula for the rotation angle follows from the fact that the trace of the matrix is equal to the sum of its eigenvalues.

The ratio of the components of the rotation axis can be found as follows: v is the rotation axis and therefore $Rv = v$. R is orthogonal and we get $R'R = E_n$, hence $v = R'v$ and therefore $(R - R')v = 0$ from which we get the ratio.

Using the mapping $R \mapsto (v_1 \sin\theta, v_2 \sin\theta, v_3 \sin\theta, \cos\theta)$ where $v = (\, v_1 \quad v_2 \quad v_3 \,)$ is the rotation axis and θ the rotation angle of R we see that $SO(3)$ can be identified with the unit sphere in $\mathbf{R}^4$ where antipodal points are identified.

Finally we mention the following theorem about normal subgroups of $SO(3)$:

Theorem 2.29 $SO(3)$ is simple, i.e. it has only trivial normal subgroups.

2.3.3 Invariant Integration

In section 2.1.3 we sketched very briefly one way to construct a generalized Riemann integral in $\mathbf{R}^n$. In the following we need however an integral on arbitrary locally compact topological groups. In this section we will introduce such an integral. In contrast to the previous generalization this time we will not construct the integral. Instead we will list some of its properties and then we will formulate a theorem that ensures the existence of such an integral.

Definition 2.46 Let X be a topological space. Then we define:

1. The *support* of a real valued function $f : X \to \mathbf{R}$ is the closure of the set of all $x \in X$ where f does not vanish. We write:

$$\text{support}(f) = \overline{\{x \in X | f(x) \neq 0\}}\,.$$

2. $C_0(X)$ is the space of all continuous, real functions on X with compact support.
3. $C_0^+(X)$ is the subset of the non-negative functions in $C_0(X)$.
4. A real-valued function on $C_0^+(X)$ is called a *functional.* A functional is thus a mapping that maps a function to a real number.

Now we assume that the space X is a locally compact group G. We consider the transformation of functions f on G under the group operation:

Definition 2.47 1. For a function $f : G \to \mathbf{R}$ and a fixed element $a \in G$ we define the *left-translate* ${}^af$ and the *right-translate* f^a as:

$$ {}^af(x) = f(ax) \qquad f^a(x) = f(xa) $$

2. A functional I is called *left-invariant, (right-invariant)* if $I({}^af) = I(f)$ ($I(f^a) = I(f)$) for all $a \in G$ and all $f \in C_0^+(G)$.
3. A functional I is called *non-negative* if $I(f) \geq 0$ for all $f \in C_0^+$.
4. A functional I is called *positive-homogeneous* if $I(\lambda f) = \lambda I(f)$ for all $f \in C_0^+(G)$ and all $\lambda \geq 0$.

For functionals we have the following existence and uniqueness properties:

Theorem 2.30 1. Let G be a locally compact topological group. Then there exists a non-trivial (i.e. not identically zero), non-negative, left-invariant, positive homogeneous, additive functional on $C_0^+(G)$.

2. This functional is unique up to a multiplicative constant, i.e. if I and J are functionals with the above mentioned properties, then there is a constant c such that $I(f) = cJ(f)$ for all $f \in C_0^+(G)$.
3. There is also a right-invariant functional with the same properties.

Definition 2.48 The integral constructed in the previous theorem is called the *Haar Integral* of the group.

These invariant functionals on $C_0^+(G)$ can be extended to real functions in $C_0(G)$ and complex functions:

Theorem 2.31 1. The left and right-invariant functionals can be extended to functionals on $C_0(G)$ with the following construction: For an $f \in C_0(G)$ we define the positive part f^+ of f as:

$$ f^+(g) = \begin{cases} f(g) & \text{if } f(g) \geq 0 \\ 0 & \text{else} \end{cases} $$

In a similar way one can define f^-, the negative part of f. For a $f \in C_0(G)$ we define: $I(f) = I(f^+) - I(f^-)$.

2. For complex valued functions $f = f_r + if_i$ we define $I(f) = I(f_r) + iI(f_i)$.

2.3.4 Examples of Haar Integrals

We will now derive Haar integrals for some important groups. First we give a general theorem that describes the Haar integral with the help of the Jacobian.

Theorem 2.32 Assume that $X \subset \mathbf{R}^n$ is an open subset of $\mathbf{R}^n$ and that it satisfies the following conditions:

1. Assume that there is a product $\circ$ on X under which X becomes a topological group. We denote the function describing the product by $f: \mathbf{R}^{2n} \to \mathbf{R}^n; (x, y) \mapsto x \circ y$
2. Assume p_i is the projection to the coordinate axis: $p_i(x_1, ..., x_n) = x_i$. Then we assume that $f_i = p_i \circ f$ is continuously differentiable.
3. The Jacobians $J(l_a)$ and $J(r_a)$ of the left- and right-translations: $l_a(x) = f(a, x)$ and $r_a(x) = f(x, a)$ are constant.

Under these conditions the left and right Haar integrals are given by the functionals:

$$I_l(f) = \int_X f(x) \left|J(l_x)\right|^{-1} \, dx$$

$$I_r(f) = \int_X f(x) \left|J(r_x)\right|^{-1} \, dx$$

This follows directly from the chain rule.

The following examples of Haar integrals can be computed with this theorem:

Examples 2.9 1. Set $X = \mathbf{R}_0$ and define $f(x, y) = xy$. The Jacobian is given by $J(a) = a$. The left and right Jacobian are identical. Left and the right Haar integral are therefore also identical and we have

$$\int_{\mathbf{R}_0} f(g) \, dg = \int_{\mathbf{R}_0} \frac{f(x)}{|x|} \, dx$$

2. Now use the multiplicative complex group instead of the reals We get $f(\xi, \eta) = f(a + ib, x + iy) = \xi\eta = ax - by + i(ay + bx)$. The Jacobian is given by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and we find for the Haar integral:

$$\int_{\mathbf{C}_0} f(g) \, dg = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x + iy)}{x^2 + y^2} \, dx dy$$

3. Define X to be the set of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with a positive real number x and arbitrary real y. This set is a locally compact group under the usual matrix multiplication. For a fixed element $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ we find

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & bx + y \\ 0 & 1 \end{pmatrix}$$

and
$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay+b \\ 0 & 1 \end{pmatrix}$$
and the Jacobians are
$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$
We get therefore $|J(l_a)| = a^2$ and $|J(r_a)| = a$. The left- and the right Haar integral are thus different and given by
$$\int_X \frac{f(x,y)}{x^2}\, dxdy \quad \text{and} \quad \int_X \frac{f(x,y)}{x}\, dx$$
respectively.

4. In the Euler angles the invariant integral on $SO(3)$ is given by
$$\int_{SO(3)} f(g)\, dg = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\phi,\theta,\psi)\sin\theta d\phi d\theta d\psi$$

A derivation of the last integral can be found in [15].

2.3.5 Exercises

Exercise 2.22 Show that $GL(n,\mathbf{R})$ and $GL(n,\mathbf{C})$ are open sets in $\mathbf{R}^{n^2}$ and $\mathbf{C}^{n^2}$ respectively.

Exercise 2.23 A neighborhood $U(e)$ of the identity is called a *symmetrical neighborhood* if $U(e)^{-1} = U(e)$. Show that every neighborhood of the identity contains a symmetrical neighborhood. (Hint: $g \mapsto g^{-1}$ is a homeomorphism.)

Exercise 2.24 Show that every neighborhood $U(e)$ of the identity contains a neighborhood $V(e)$ such that $V(e)V(e) \subset U(e)$. (Hint: $f(gh) = gh$ is continuous.)

Exercise 2.25 Denote by $S(a,b)$ the matrix $\begin{pmatrix} \overline{b} & -\overline{a} \\ a & b \end{pmatrix}$ and by $R(a,b)$ the matrix:
$$\begin{pmatrix} (b^2 - a^2 + \overline{b}^2 - \overline{a}^2)/2 & i(b^2 + a^2 - \overline{b}^2 - \overline{a}^2)/2 & ab + \overline{ab} \\ i(-\overline{b}^2 + \overline{a}^2 + b^2 - a^2)/2 & (b^2 + a^2 + \overline{b}^2 + \overline{a}^2)/2 & i(ab - \overline{ab}) \\ -a\overline{b} - \overline{a}b & i(\overline{a}b - a\overline{b}) & b\overline{b} - a\overline{a} \end{pmatrix}$$

Show the the mapping $f : SU(2) \to SO(3)$ defined as $f(S(a,b)) = R(a,b)$ is an open continuous homomorphism of $SU(2)$ onto $SO(3)$. Show that the kernel is given by the matrices E_2 and $-E_2$.

Hint: Investigate first
$$f\left(\begin{pmatrix} e^{-(i\varphi)/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\right)$$
and
$$f\left(\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}\right)$$

Exercise 2.26 Assume V is a finite-dimensional, real vector space and $\langle\ ,\ \rangle : V \times V \to \mathbf{R}$ is a bilinear map. Define

$$G = \{A | A : V \to V, \langle v, w\rangle = \langle Av, Aw\rangle \text{ for all } v, w \in V\}.$$

- Show that G is a group.
- Assume $V = \mathbf{R}^n, H = \mathbf{R}$ and $\langle\ ,\ \rangle$ is the standard scalar product. What is G in this case?

Exercise 2.27 The 2-D rotations transform circles into circles, the n-dimensional rotations map n-dimensional spheres into themselves. The circles and spheres are thus invariant under the elements of $SO(2)$ and $SO(n)$ respectively.

Now consider the vector space $V = \mathbf{R}^2, H = \mathbf{R}$ and the mapping

$$\langle x, y\rangle = x_1 y_1 - x_2 y_2.$$

Assume further that L is a function $L : V \to V$ that leaves this mapping invariant: $\langle v, w\rangle = \langle Lv, Lw\rangle$. What can be said of the invariant subspaces of L?

(Remark: Transformations that leave functions of the form $\langle x, y\rangle = \sum_{k=1}^{n-1} x_k y_k - x_n y_n$ invariant are called *Lorentz groups.*)