Chapter 4

Representations of some Matrix Groups

In this chapter we will construct representations for a number of important groups. In the first section we will treat SO(3) in some detail. We do this since this group is of great practical importance and since it is the simplest compact, non-commutative group. By constructing the representations of SO(3) we will also demonstrate the usage of some of the fundamental tools of Lie group theory. We conclude the section on SO(3) by collecting a number of formulas that are important in 3-D image processing. Further practical information about the surface harmonics can be found in books on special functions like [11].

In the next two sections we will then consider the group of motions in the euclidian plane and the special linear group of 2 × 2 matrices. In the case of the group $M(2)$ we construct some irreducible representations of this group using the same techniques as in the SO(3) case.

A complete theory of representations for these groups is considerably more difficult then the representation theory for SO(3) and lies beyond the scope of these lecture notes. We will therefore only give a collection of basic facts about these groups. The interested reader can find detailed investigations of these groups in the literature. We mention here that a detailed treatment of the representation theory of the euclidian motion group $M(2)$ can be found in [45] (chapter IV, pages 149-198) and the representations of $SL(2,\mathbf{C})$ are investigated in [14] (chapter IV, pages 202-272).

4.1 The Representations of $SO(3)$

From the previous sections it should be clear that it is important to know a complete set of representations of a group. In this section we will now construct such a set for the group of rotations in 3-D. This is considerably more difficult than in the 2-D case since $SO(3)$ is not commutative. We can therefore expect that we will have to consider higher dimensional representations. Since $SO(3)$ is compact it is however only necessary to find all irreducible, unitary representations (see 3.11 and 3.10). We will now derive a complete set of such representations.The representation space is $L^2(S^2)$ the space of square integrable functions on the unit sphere.

This will lead us to a system of functions Y_l^m that is connected to the three-dimensional representations as the exponential functions are connected to the two-dimensional rotation group $SO(2)$.

Recall that the exponential functions are characterized by the functional equation

$$e^{in(\phi+\psi)} = e^{in\phi} \cdot e^{in(\psi)} \tag{4.1}$$

and

$$e^{2n\pi i} = 1.$$

In the group theoretical interpretation this means that the value of the transformed function $e^{in(\phi+\psi)}$ is equal to the original function $e^{in\phi}$ times a complex factor $e^{in\psi}$. In the three-dimensional case we have as the domain the unit sphere $\mathcal{S}^2$ and the rotations $R \in SO(3)$ transform a function on the sphere by rotating the sphere first: $f^R(X) = f(R^{-1}X)$.

If we have a finite dimensional subspace of $L^2(\mathcal{S}^2)$ that is invariant under this action of $SO(3)$ then we can select a basis $Y^1, ..., Y^m$ of this subspace and we get for the transformed functions $Y^k(R^{-1}X)$ the equation:

$$Y^k(R^{-1}X) = \sum_{l=1}^{m} t_{lk}(R)Y^l(X)$$

Writing this in matrix form gives:

$$Y(R^{-1}X) = T(R)Y(X) \tag{4.2}$$

Equations 4.1 and 4.2 describe the same type of transformation with the only difference that we have replaced the function itself with a vector of functions and the complex factor becomes a complex matrix. The functions Y_l^m have thus the same transformation property as the complex exponentials. This is the more practical side of this chapter. On the more theoretical side this chapter demonstrates some of the basic techniques from Lie theory.

In this chapter we follow closely the book of Gelfand et al. [15]. For more information on Lie theory the reader may consult one of the books on Lie theory (for example [47] or [55]).

4.1.1 Exponential of a Matrix

Let A be any $n \times n$ matrix with elements a_{ij} and assume that these elements have a common upper bound $|a_{ij}| \leq C$. Since A is a square matrix we can compute A^m for all integers m. If we denote the elements of A^m by $a_{ij}^{(m)}$ then we find with a simple computation:

$$\left|a_{ij}^{(m)}\right| \leq (nC)^m$$

From this inequality we see that the following definition is meaningful:

Definition 4.1

$$e^A = \sum_{m=0}^{\infty} \frac{1}{m!} A^m$$

We call e the *exponential function.*

For the exponential function we collect a number of useful properties in the next theorem:

Theorem 4.1
1. If B is an invertible $n \times n$ matrix then we have
$$e^{BAB^{-1}} = Be^A B^{-1}$$
2. If $\lambda_i (i = 1, \dots n)$ are the eigenvalues of A then the eigenvalues of e^A are given by e^{λ_i}.
3. $\det(e^A) = e^{\mathrm{tr} A}$
4. e^A is regular for all matrices A.
5. If $AB = BA$ then $e^{A+B} = e^A e^B$.

Most of these properties are simple consequences of the definition of the exponential function; the second part of the theorem about the eigenvalues is proved by induction on the number of eigenvalues (for a proof of this theorem and the next two see [5]).

Theorem 4.2 Let $t \in \mathbf{R}$ be a real variable and A be a fixed $n \times n$ matrix. The mapping $e^{\cdot}\mathbf{R} \to GL(n, \mathbf{C}); t \mapsto e^{tA}$ is a continuous homomorphism of the additive group of reals into $GL(n, \mathbf{C})$.

Theorem 4.3 There is a neighborhood U of the zero matrix in the space of all complex $n \times n$ matrices that is homeomorphic to a neighborhood V of E in $GL(n, \mathbf{C})$. The homeomorphism is given by $A \mapsto e^A$.

4.1.2 Infinitesimal Rotations

In this section we investigate an irreducible, unitary representation of $SO(3)$ as a matrix function parametrized by the rotation axis and the rotation angle. We show that this highly nonlinear function can be completely described by three matrices.

Suppose that T is an irreducible, unitary representation of $SO(3)$ and that we selected a fixed orthonormal basis in the representation space. Then T is completely described by a unitary matrix. This matrix will also be denoted by T. We describe a rotation g by the vector (ξ_1, ξ_2, ξ_3) where the direction of the vector is given by the rotation axis and the length of the vector is equal to the rotation angle. The identity rotation is described by the zero vector. Then T becomes a function of the ξ_i and we have $T(0,0,0) = E$ where E is the identity matrix. It can be proved that T as a function of the parameters ξ_i is infinitely differentiable and we can thus develop T into a Taylor series:

$$T(\xi_1, \xi_2, \xi_3) = E + \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3 + \dots \tag{4.3}$$

This is a matrix equation with constant matrices A_i. To see what these matrices are we consider the case of a rotation around the x-axis with rotation angle ξ. In this case we have:

$$T(\xi, 0, 0) = E + \xi A_1 + \dots \tag{4.4}$$

and we find that

$$A_1 = \lim_{\xi \to 0} \frac{T(\xi,0,0) - T(0,0,0)}{\xi} = \frac{dT(\xi_1, \xi_2, \xi_3)}{d\xi 1} \tag{4.5}$$

These matrices are thus the partial derivatives of T at the point $(0,0,0)$ and we define:

Definition 4.2 The matrices A_i in equation 4.3 are called the *matrices of infinitesimal rotations about the coordinate axes.*

In the next section we will describe the possible forms of the $A_i's$ but now we will show in the next theorem that T is completely defined by these three matrices A_i.

Theorem 4.4 The matrices A_i determine completely the representation, i.e. given the A_i we can determine $T(\xi_1, \xi_2, \xi_3)$ for all ξ_i.

To see this take an arbitrary vector $(\xi_1, \ \xi_2, \ \xi_3)$ and two rotations $g(t\xi_1, \ t\xi_2, \ t\xi_3)$ and $g(s\xi_1, \ s\xi_2, \ s\xi_3)$. These are rotations with the common rotation axis given by the vector $(\xi_1, \ \xi_2, \ \xi_3)$ and rotation angles $t\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$ and $s\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$ respectively. The product of these two rotations is the rotation

$$\begin{aligned} & g((s+t)\xi_1, \ (s+t)\xi_2, \ (s+t)\xi_3) = \\ = \ & g(t\xi_1, \ t\xi_2, \ t\xi_3)\, g(s\xi_1, \ s\xi_2, \ s\xi_3) \end{aligned} \tag{4.6}$$

Since T is a representation we find:

$$\begin{aligned} & T((s+t)\xi_1, \ (s+t)\xi_2, \ (s+t)\xi_3) = \\ = \ & T(s\xi_1, \ s\xi_2, \ s\xi_3)\, T(t\xi_1, \ t\xi_2, \ t\xi_3) \end{aligned} \tag{4.7}$$

We differentiate both side of equation 4.7 with respect to s and set $s = 0$ to get the differential equation:

$$\frac{d}{dt}T(t\xi_1, t\xi_2, t\xi_3) = \frac{d}{ds}T(s\xi_1, s\xi_2, s\xi_3)|_{s=0}T(t\xi_1, t\xi_2, t\xi_3) \tag{4.8}$$

From the Taylor expansion 4.3 we find

$$\frac{d}{ds}T(s\xi_1, s\xi_2, s\xi_3)|_{s=0} = A_1\xi_1 + A_2\xi_2 + A_3\xi_3 \tag{4.9}$$

If we set $X(t) = T(t\xi_1, t\xi_2, t\xi_3)$ then we get the differential equation for X:

$$\frac{d}{dt}X(t) = (A_1\xi_1 + A_2\xi_2 + A_3\xi_3)X(t) \tag{4.10}$$

The solution that obeys the initial value $X(0) = E$ is given by

$$X(t) = \mathrm{e}^{t(A_1\xi_1 + A_2\xi_2 + A_3\xi_3)} \tag{4.11}$$

or

$$T(\xi_1, \xi_2, \xi_3) = \mathrm{e}^{A_1\xi_1 + A_2\xi_2 + A_3\xi_3} = \mathrm{e}^{A_\xi} \tag{4.12}$$

where we used the abbreviation $A_\xi = A_1\xi_1 + A_2\xi_2 + A_3\xi_3$.

This shows that the representation is completely specified by the matrices A_i.

4.1.3 The Commutation Relations

In this section we investigate the relations between the infinitesimal rotations A_i.

If g_0 is a fixed rotation and g is another rotation then we find that the rotations $\tilde{g}_0 = gg_0g^{-1}$ and G_0 have the same rotation angle. This follows from the fact that $\tilde{g}_0$ and g_0 have the same eigenvalues and from the connection between the rotation angle and the eigenvalues. Now we describe the rotation g_0 by the vector η as defined in the previous section. Applying the rotation g to this vector η gives a new vector $\tilde{\eta} = g\eta$. η describes the rotation axis of g_0 and we have therefore $g_0\eta = \eta$. From this we get $\tilde{g}_0\tilde{\eta} = gg_0g^{-1}\tilde{\eta} = gg_0\eta = \tilde{\eta}$. Since g_0 and $\tilde{g}_0$ have the same rotation angle we find that $\tilde{\eta}$ describes the rotation $\tilde{g}_0$.

For notational convenience we write now T_g instead of $T(g)$. From the representation property we find:

$$T_{\tilde{g}_0} = T_{gg_0g^{-1}} = T_gT_{g_0}T_{g^{-1}}.$$

Inserting this into the Taylor series gives:

$$T_{\tilde{g}_0} = \mathrm{e}^{A_{\tilde{\eta}}} = T_{gg_0g^{-1}} = T_gT_{g_0}T_{g^{-1}} = T_g\mathrm{e}^{A_\eta}T_{g^{-1}} \tag{4.13}$$

Using theorem 4.1 we get:

$$T_gA_\eta T_{g^{-1}} = A_{\tilde{\eta}} \tag{4.14}$$

Now select as g a rotation with angle α around the x–axis, i.e. $T_g = T(\alpha, 0, 0) = \mathrm{e}^{A_1\cdot\alpha}$ and $\eta = (\,0, \quad 1, \quad 0\,)$. Since $\tilde{\eta} = g\eta$ we have also $\tilde{\eta} = (\,0, \quad \cos\alpha, \quad \sin\alpha\,)$. Expanding the exponential function we find therefore from 4.14 and $T_{g^{-1}} = T(-\alpha, 0, 0)$:

$$A_2 + \alpha(A_1A_2 - A_2A_1) + \alpha^2 ... = \cos\alpha A_2 + \sin\alpha A_3 \tag{4.15}$$

and from this we get $A_1A_2 - A_2A_1 = A_3$

Definition 4.3 If A and B are two square-matrices then the expression $AB - BA$ is called the *commutator* of A and B. We write:

$$[A, B] = AB - BA \tag{4.16}$$

This expression is sometimes also called the *bracket* of A and B.

We find by similar computations the following theorem:

Theorem 4.5 If $T : g \to T(g)$ is an arbitrary representation of $SO(3)$ and A_1, A_2 and A_3 are the matrices corresponding to the infinitesimal rotations about the coordinate axes then we have:

$$\begin{aligned} [A_1, A_2] &= A_3 \\ [A_2, A_3] &= A_1 \\ [A_3, A_1] &= A_2 \end{aligned} \tag{4.17}$$

There is the following relation between the commutator and the vector product (see [15]):

Theorem 4.6 For a vector $(\,\xi_1, \quad \xi_2, \quad \xi_3\,)$ we define as usual the matrix $A_\xi = A_1\xi_1 + A_2\xi_2 + A_3\xi_3$ with the infinitesimal matrices A_i. Then we have the following relation between the bracket and the vector product: if ξ, η are two 3-D vectors and $\zeta = \xi \times \eta$ is the vector product of ξ and η then we have

$$[A_\xi, A_\eta] = A_\zeta \tag{4.18}$$

4.1.4 Canonical Basis of an Irreducible Representation

In the next theorem we describe what consequences the unitary constraint has on the matrices A_i. T was a unitary representation and the matrices T_g are therefore unitary matrices satisfying: $T_g^\star T_g = E$. If we choose g as the rotation around the x-axis then we find: $T(\xi,0,0)^\star T(\xi,0,0) = E$ (where $T^\star$ is the conjugate complex transpose of T). Inserting the Taylor expansion for T and comparing the linear entries we find

$$A_1 + A_1^\star = 0 \tag{4.19}$$

or $A_1 = -A_1^\star$ and therefore:

Theorem 4.7 Define $H_k = iA_k$, then we have

1. H_k are Hermitian matrices, i.e. $H_k = H_k^\star$
2. The H_k satisfy the following relations:

$$\begin{aligned} [H_1, H_2] &= iH_3 \\ [H_2, H_3] &= iH_1 \\ [H_3, H_1] &= iH_2 \end{aligned} \tag{4.20}$$

In the next theorem we introduce two new matrices which will be more convenient in the following computations.

Theorem 4.8 If the operators H_+, H_- are defined as:

$$\begin{aligned} H_+ &= H_1 + iH_2 \\ H_- &= H_1 - iH_2 \end{aligned} \tag{4.21}$$

then we have:

1. $$H_+^\star = H_- \tag{4.22}$$
2. The matrices H_+, H_-, H_3 satisfy the following relations:

$$\begin{aligned} [H_+, H_3] &= -H_+ \\ [H_-, H_3] &= H_- \\ [H_+, H_-] &= 2H_3 \end{aligned} \tag{4.23}$$

We now proceed as follows: First we find all matrices H_+, H_-, H_3 that satisfy the conditions of theorem 4.8, then we use 4.21 to find the matrices H_1, H_2 and H_3. These in turn are then used in theorem 4.7 to find the matrices A_k which define the representation.

For the eigenvectors of H_3 we find the following theorem:

Theorem 4.9 Let f be an eigenvector of H_3 with eigenvalue λ:

$$H_3 f = \lambda f$$

and define:

$$\begin{aligned} f_+ &= H_+ f \\ f_- &= H_- f \end{aligned}$$

then we find:

1. f_+ is either the zero-vector or an eigenvector of H_3 with eigenvalue $\lambda + 1 : H_3 f_+ = (\lambda + 1) f_+$
2. f_- is either the zero-vector or an eigenvector of H_3 with eigenvalue $\lambda - 1 : H_3 f_- = (\lambda - 1) f_-$

Definition 4.4 The operators H_+ and H_- are called the *raising and lowering operators.*

We show only the first part concerning f_+ :

$$\begin{aligned} H_3 f_+ = H_3 H_+ f = [H_3, H_+] f + H_+ H_3 f = \\ H_+ f + H_+ \lambda f = (\lambda + 1) H_+ f = (\lambda + 1) f_+ \end{aligned} \tag{4.24}$$

Now we proceed to construct H_+, H_-, H_3 : H_3 is Hermitian and its eigenvalues are therefore real. Let l be the largest eigenvalue with (normalized) eigenvector f_l. If $H_- f_l \neq 0$ then we define the unit vector f_{l-1} by $H_- f_l = \lambda_l f_{l-1}$. From the previous theorem 4.9 we find that f_{l-1} is also an eigenvector of H_3 with eigenvalue $l - 1$. Continuing this process we find a series of eigenvectors $f_l, f_{l-1}, f_{l-2}, \ldots$ of H_3 with eigenvalues $l, l-1, l-2, \ldots$. This process must stop after finitely many steps, say with the index $k : H_- f_k = 0$. We thus constructed a series of vectors f_m that satisfy the following conditions:

$$H_3 f_m = m f_m \tag{4.25}$$

and

$$H_- f_m = \lambda_m f_{m-1} \tag{4.26}$$

$H_+ f_m$ is either the zero vector or an eigenvector of H_3 and since f_l belongs to the largest eigenvalue we have $H_+ f_l = 0$. For $H_+ f_{l-1}$ we get:

$$H_+ f_{l-1} = \frac{1}{\lambda_l} H_+ H_- f_l = \frac{1}{\lambda_l} [H_+, H_-] f_l + \frac{1}{\lambda_l} H_- H_+ f_l = \frac{2}{\lambda_l} H_3 f_l = \frac{2l}{\lambda_l} f_l. \tag{4.27}$$

We define β_l by: $H_+ f_{l-1} = \beta_l f_l$ and we find for the other f_m by induction:

$$\begin{aligned} H_+ f_m &= \frac{1}{\lambda_{m+1}} H_+ H_- f_{m+1} \\ &= \frac{1}{\lambda_{m+1}} [H_+, H_-] f_{m+1} + \frac{1}{\lambda_{m+1}} H_- H_+ f_{m+1} \\ &= \frac{2}{\lambda_{m+1}} H_3 f_{m+1} + \frac{\beta_{m+2}}{\lambda_{m+1}} H_- f_{m+2} \end{aligned} \tag{4.28}$$

Using equation 4.26 we get

$$H_+ f_m = \frac{2(m+1) + \lambda_{m+2} \beta_{m+2}}{\lambda_{m+1}} f_{m+1} \tag{4.29}$$

and if we set

$$\beta_{m+1} = \frac{2(m+1) + \lambda_{m+1} \beta_{m+2}}{\lambda_{m+1}} \tag{4.30}$$

then equation 4.29 becomes

$$H_+ f_m = \beta_{m+1} f_{m+1}. \tag{4.31}$$

Using equations 4.26 and 4.31 and $H_+^* = H_-$ from equation 4.22 we get:

$$\beta_m \langle f_m, f_m \rangle = \langle H_+ f_{m-1}, f_m \rangle = \langle f_{m-1}, H_- f_m \rangle = \lambda_m \langle f_{m-1}, f_{m-1} \rangle \tag{4.32}$$

The eigenvectors are normalized and we find $\lambda_m = \beta_m$.

Using this relation between the λ's and β's in equation 4.30 we find $\lambda_m^2 - \lambda_{m+1}^2 = 2m$ and by a summation:

$$\lambda_m^2 = \lambda_m^2 - \lambda_{l+1}^2 = 2l + 2(l-1) + \cdots + 2m = (l+m)(l-m+1) \tag{4.33}$$

Now k was the index of the lowest eigenvector and therefore $\lambda_k = 0$. Consequently $0 = \lambda_k^2 = (l+k)(l-k+1)$ and therefore $k = -l$. l was the largest eigenvalue, $-l$ the lowest and the eigenvalue was decremented by one in each processing step. It follows that the number of eigenvectors is $2l+1$ and l is therefore an integer or half an odd integer.

Under H_+, H_- and H_3 an eigenvector f_m is carried into another eigenvector and since we assumed that the representation is irreducible we see that the constructed space of eigenvectors of H_3 is the whole representation space. We collect this result in the following theorem:

Theorem 4.10 For any irreducible representation the transformations H_+, H_- and H_3 define an orthogonal basis consisting of the normalized eigenvectors f_m of H_3. These eigenvectors satisfy the following conditions:

$$\begin{aligned} H_+ f_m &= \lambda_{m+1} f_{m+1} \\ H_- f_m &= \lambda_m f_{m-1} \\ H_3 f_m &= m f_m. \end{aligned} \tag{4.34}$$

We have $m = -l, -l+1, \ldots, l$ with an integer or half an odd integer l and for the constants λ_m we find: $\lambda_m = \sqrt{(l+m)(l-m+1)}$

Definition 4.5 The basis $f_{-l}, \ldots, f_l$ of the normalized eigenvectors of H_3 is called the *canonical basis of the representation* .

Returning to our initial problem of finding the possible matrices of the infinitesimal rotations A_k we have:

Theorem 4.11 Assume that $\lambda_m = \sqrt{(l+m)(l-m+1)}$, that f_m is the canonical basis and that $m = -l, -l+1, \ldots, l$. Then we have:

1. Each irreducible representation of $SO(3)$ is defined by an integer or half an odd integer l. The matrices A_k are given by the conditions:

$$\begin{aligned} A_1 f_m = -iH_1 f_m &= -\frac{i}{2}(H_+ + H_-) f_m = -\frac{i}{2}\lambda_{m+1} f_{m+1} - \frac{i}{2}\lambda_m f_{m-1} \\ A_2 f_m = -iH_2 f_m &= -\frac{1}{2}(H_+ - H_-) f_m = -\frac{1}{2}\lambda_{m+1} f_{m+1} + \frac{1}{2}\lambda_m f_{m-1} \\ A_3 f_m = -iH_3 f_m &= -im f_m \end{aligned} \tag{4.35}$$

2. The matrices have the following form:

$$\begin{aligned}
A_1 &= -\frac{i}{2}\begin{pmatrix} 0 & \lambda_{-l+1} & 0 & \dots & 0 & 0 \\ \lambda_{-l+1} & 0 & \lambda_{-l+2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \lambda_l \\ 0 & 0 & 0 & \dots & \lambda_l & 0 \end{pmatrix} \\
A_2 &= \frac{1}{2}\begin{pmatrix} 0 & \lambda_{-l+1} & 0 & \dots & 0 & 0 \\ -\lambda_{-l+1} & 0 & \lambda_{-l+2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \lambda_l \\ 0 & 0 & 0 & \dots & -\lambda_l & 0 \end{pmatrix} \\
A_3 &= \begin{pmatrix} il & 0 & 0 & \dots & 0 & 0 \\ 0 & i(l-1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -i(l-1) & 0 \\ 0 & 0 & 0 & \dots & 0 & -il \end{pmatrix}
\end{aligned} \tag{4.36}$$

Definition 4.6 Each irreducible representation of $SO(3)$ is uniquely defined by the number l given in the last theorem 4.11. l is called the *weight* of the representation.

Up to now we have shown that for an irreducible representation T the matrices T_g are of the form $T_g = T(\xi_1, \xi_2, \xi_3) = T(\xi) = e^{A\xi}$ (see equation 4.12) and that the matrices A_k are given by the conditions in theorem 4.11. However, we have not shown that there is such a representation for each l. We will later construct such a representation. Now we will only show that if there is a representation of the form given in equation 4.12 with the A_k satisfying the conditions in theorem 4.11 then this representation is irreducible.

We show that every invariant subspace has dimension $2l+1$. Consider an invariant subspace and denote the largest eigenvector of H_3 in this subspace by $h = \sum_{m=-l}^{l} c_m f_m$. From theorem 4.9 we find that H_+h is the zero vector and from theorem 4.10 we get

$$0 = H_+h = \sum_{m=-l}^{l} c_m H_+ f_m = \sum_{m=-l}^{l} c_m \lambda_{m+1} f_m$$

The f_m are independent and therefore $c_m\lambda_{m+1} = 0$ for all m. If $m < l$ then $\lambda_{m+1} \neq 0$ and therefore $c_m = 0$ and from this we see that f_l is an element of the invariant subspace. Since the space is invariant under H_- we find that also all the other f_m are elements of the invariant subspace.

In the next theorem we introduce still another transformation and we show how it can be used to characterize the representation space.

Theorem 4.12 Define the transformation H^2 as

$$H^2 = H_1^2 + H_2^2 + H_3^2 \tag{4.37}$$

Then we have:

1. $[H^2, H_1] = [H^2, H_2] = [H^2, H_3]$

2. All vectors f in the representation space of a representation of weight l satisfy the equation:
$$H^2 f = l(l+1)f \tag{4.38}$$

The first part of the theorem is an easy calculation using the definitions of the operators involved and the properties of the bracket. To see the second part check first that $H_+H_- = H_1^2 + H_2^2 + H_3$ and find that $H^2 = H_+H_- - H_3 + H_3^2$ then from theorem 4.10 deduce

$$H_+H_-f_m = \lambda_m^2 f_m$$

$$H_3 f_m = m f_m$$

$$H_3^2 f_m = m^2 f_m$$

But $\lambda_m = \sqrt{(l+m)(l-m+1)}$ and therefore we get:

$$H^2 f_m = (\lambda_m^2 - m + m^2) f_m = l(l+1) f_m$$

4.1.5 Spherical Functions

In the last section we saw how to characterize a given irreducible representation. In this section we will now actually construct these representations. For this purpose we consider the function space L of differentiable functions on S^2, the unit sphere in three-dimensional space. On S^2 we introduce polar coordinates φ, θ defined by the equations:

$$\begin{aligned} x &= \cos\varphi \sin\theta \\ y &= \sin\varphi \sin\theta \\ z &= \cos\theta. \end{aligned} \tag{4.39}$$

or

$$\begin{aligned} \varphi &= \arctan\frac{y}{x} \\ \theta &= \arctan\frac{\sqrt{x^2+y^2}}{z}. \end{aligned} \tag{4.40}$$

The scalar product in this space is given by

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\overline{g(\theta,\varphi)} \sin\theta \, d\theta d\varphi = \int_{S^2} f\bar{g}\, d\omega \tag{4.41}$$

As usual we define the representation T by $T(g)f(X) = f(g^{-1}(X))$. This is indeed a representation as can be seen by checking that $T(g)f$ is indeed differentiable. We define

Definition 4.7 1. The functions on the sphere that belong to an irreducible representation of weight l are called the *spherical functions of order l.*

2. The functions $f_m(\theta,\varphi)$ forming the canonical basis in the space of spherical functions of order l are called the *basic spherical functions of the l-th order.*

3. The basic spherical functions of the l-th order will be denoted by $Y_l^m(\theta,\varphi)$

We will now derive these basic functions Y_l^m. In the last sections we saw that the canonical basis was defined with the help of the matrices A_k of the infinitesimal rotations and we must therefore first obtain the transformations which correspond to these infinitesimal rotations.

We first consider A_3 describing an infinitesimal rotation around the z-axis:

Theorem 4.13

$$A_3 f = -\frac{\partial f(\theta,\varphi)}{\partial\varphi}$$

To see this take a rotation g around the z-axis with rotation angle α. Then we have $T(g) = \mathrm{e}^{\alpha A_3}$ and we expand therefore $T(g)f$ in terms of α :

$$T(g)f(\theta,\varphi) = f(g^{-1}(\theta,\varphi)) = f(\theta,\varphi-\alpha)$$

and from

$$f(\theta,\varphi-\alpha) = f(\theta,\varphi) - \alpha\frac{\partial f(\theta,\varphi)}{\partial\varphi} + \dots \tag{4.42}$$

we find the form of A_3. For the matrices A_1 and A_2, i.e. the rotations around the x- and y-axis we get

Theorem 4.14

$$\begin{aligned} A_1 &= \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} \\ A_2 &= -\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} \end{aligned} \tag{4.43}$$

We compute only the expression for the x-axis rotation, the other expression can be derived similarly.

If g is a rotation around the x-axis with rotation angle then we expand again $T(g)f$ in a series in α and we get then A_1 as the operator belonging to the linear term in α. We thus get:

$$A_1 = \lim_{\alpha\to 0}\frac{T(\alpha,0,0) - T(0,0,0)}{\alpha}$$

If we expand $T(\alpha,0,0)f$ into a Taylor series then we get

$$T(\alpha,0,0)f(\theta,\varphi) = f(\theta,\varphi) + \left(\frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial\alpha} + \frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial\alpha}\right)|_{\alpha=0}\,\alpha + \dots$$

If $(x', \ y', \ z')$ is the rotated coordinate system $g^{-1}(x, \ y, \ z)$ then we see that

$$\begin{aligned} \frac{\partial x'}{\partial\alpha} &= 0 \\ \frac{\partial y'}{\partial\alpha}|_{\alpha=0} &= z \\ \frac{\partial z'}{\partial\alpha}|_{\alpha=0} &= -y \end{aligned} \tag{4.44}$$

But

$$\begin{aligned}\frac{\partial x'}{\partial \alpha} &= \frac{\partial x'}{\partial \theta}\frac{\partial \theta}{\partial \alpha} + \frac{\partial x'}{\partial \varphi}\frac{\partial \varphi}{\partial \alpha} \\ &= \cos\theta\cos\varphi\frac{\partial \theta}{\partial \alpha} - \sin\theta\sin\varphi\frac{\partial \varphi}{\partial \alpha} \qquad (4.45)\end{aligned}$$

leading to

$$0 = \cos\theta\cos\varphi\frac{\partial \theta}{\partial \alpha} - \sin\theta\sin\varphi\frac{\partial \varphi}{\partial \alpha} \qquad (4.46)$$

For y' and z' we get with the same calculations:

$$\begin{aligned}\cos\theta &= \cos\theta\sin\varphi\frac{\partial \theta}{\partial \alpha} + \sin\theta\cos\varphi\frac{\partial \varphi}{\partial \alpha} \\ -\sin\theta\sin\varphi &= -\sin\theta\frac{\partial \theta}{\partial \alpha} \qquad (4.47)\end{aligned}$$

From equations 4.46 and 4.47 we find

$$\begin{aligned}\frac{\partial \theta}{\partial \alpha} &= \sin\varphi \\ \frac{\partial \varphi}{\partial \alpha} &= \cot\theta\cos\varphi \qquad (4.48)\end{aligned}$$

Inserting these expressions in the Taylor series for $T(g)f$ we get the following expression for A_1:

$$T(g)f = f(\theta,\varphi) + \left(\frac{\partial f}{\partial \theta}\sin\varphi + \frac{\partial f}{\partial \varphi}\cot\theta\cos\varphi\right)|_{\alpha=0}\ \alpha + \ldots$$

From the expressions for the operators A_k it is now easy to compute the differential operators belonging to the operators H_3, H_+, and H_-.

Theorem 4.15

$$\begin{aligned}H_+ &= e^{i\varphi}\left(\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \varphi}\right) \\ H_- &= e^{-i\varphi}\left(-\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \varphi}\right) \\ H_3 &= -i\frac{\partial}{\partial \varphi} \qquad (4.49)\end{aligned}$$

From the previous derivations we know that the basic spherical functions Y_l^m are given by the eigenfunctions of the H_3 operator, i.e. they are the solutions of the differential equation:

$$H_3 Y_l^m(\theta,\varphi) = -i\frac{\partial}{\partial \varphi}Y_l^m(\theta,\varphi) = mY_l^m(\theta,\varphi)$$

Hence we find:

$$Y_l^m(\theta,\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}F_l^m(\theta) \qquad (4.50)$$

Since Y is a one-valued function on the sphere and since $Y_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi+2\pi)$ we find that m is an integer. Since the Y are also normalized we get from equation 4.50 the following integral for the function F:

$$1 = \int_0^\pi |F_l^m(\theta)|^2 \sin\theta \, d\theta \tag{4.51}$$

Our next task is the derivation of the function F. This will be done with the help of the H^2 operator. Recall (theorem 4.12) that all functions in the representation space satisfy the equation $H^2 f = l(l+1)f$. Using theorem 4.15 we find for H^2 the *differential equation of the spherical functions of the l-th order*:

$$-H^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{4.52}$$

Substituting the expression for Y_l^m from equation 4.50 into 4.52 we find the differential equation for F_l^m:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_l^m}{\partial\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] F_l^m(\theta) = 0 \tag{4.53}$$

or after the substitution $\mu = \cos\theta, P_l^m(\mu) = F_l^m(\cos\theta)$:

$$\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial P_l^m(\mu)}{\partial\mu}\right] + \left[l(l+1) - \frac{m^2}{1-\mu^2}\right] P_l^m(\mu) = 0 \tag{4.54}$$

We summarize this in the next theorem:

Theorem 4.16 The basic spherical functions have the form:

$$Y_l^m(\theta,\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} P_l^m(\cos\theta)$$

where the P_l^m are solutions of the differential equation 4.54.

It now remains to find an expression for the dependence of Y_l^m on θ. The next theorem, together with theorem 4.16 will completely specify the form of the basis functions:

Theorem 4.17 1. The functions $P_l^m(\cos\theta)$ defined as in theorem 4.16 are given by:

$$P_l^m(\mu) = \sqrt{\frac{(l+m)!}{(l-m)!}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^l \cdot l!}\left(1-\mu^2\right)^{-m/2}\frac{d^{l-m}(\mu^2-1)^l}{d\mu^{l-m}} \tag{4.55}$$

2. If we write $P_l(\mu) = P_l^0(\mu)$ then we have:

$$P_l(\mu) = \sqrt{\frac{2l+1}{2}}\frac{1}{2^l \cdot l!}\frac{d^l(\mu^2-1)^l}{d\mu^l} \tag{4.56}$$

Definition 4.8 1. The polynomial P_l is called the *normalized Legendre polynomial of order l.*

2. The polynomial P_l^m is called the *normalized associated Legendre function.*

We consider first Y_l^l the solution of the equations:

$$\begin{aligned} H_3 Y_l^l &= l Y_l^l \\ H_+ Y_l^l &= 0 \end{aligned} \tag{4.57}$$

We saw in equation 4.50 that the function $Y_l^l(\theta,\varphi)$ is of the form $Y_l^l(\theta,\varphi) = \frac{1}{\sqrt{2\pi}} e^{il\varphi} F_l^l(\theta)$. From the form of H_+ (theorem 4.15) and the second equation in 4.57 we get:

$$\frac{dF_l^l(\theta)}{d\theta} - l \cot\theta F_l^l(\theta) = 0 \tag{4.58}$$

The general solution of this differential equation is given by

$$F_l^l(\theta) = C \sin^l \theta \tag{4.59}$$

Using the normalizing condition in equation 4.51 we find for the constant C the value

$$C = (-1)^l \frac{1}{2^l l!} \sqrt{\frac{2l+1}{2}} \sqrt{(2l)!} \tag{4.60}$$

Having found the form of Y_l^l we use the equation $H_- Y_l^m = \lambda_m Y_l^{m-1}$ to get the expression for the other functions. Using the expression form H_- we get

$$e^{-i\varphi}\left(-\frac{\partial Y_l^m}{\partial\theta} + i\cot\theta \frac{\partial Y_l^m}{\partial\varphi}\right) = \lambda_m Y_l^{m-1} \tag{4.61}$$

Inserting 4.50 and canceling factors depending only on φ we get the following recurrence relation for F_l^m :

$$-\frac{dF_l^m(\theta)}{d\theta} - m\cot\theta F_l^m(\theta) = \lambda_m F_l^{m-1}(\theta) \tag{4.62}$$

Making again the substitution $\mu = \cos\theta$ and $P_l^m(\mu) = F_l^m(\theta)$ gives:

$$\sqrt{1/\mu^2}\left(\frac{dP_l^m(\mu)}{\mu} - m\frac{\mu}{1-\mu^2}P_l^m(\mu)\right) = \lambda_m P_l^{m-1}(\mu) \tag{4.63}$$

Using this recurrence relation between the P_l^m it is possible to derive the required expression for the functions P_l^m.

4.1.6 Homogeneous Polynomials

Before we leave the group $SO(3)$ we will present a more traditional approach to surface harmonics. We first derive a transformation formula that describes explicitly how the surface harmonics transform under a rotation. This is important in the pattern recognition applications of representation theory for the following reason.

Assume we have a fixed pattern $p_0(x,y,z)$ like an edge or a line in 3-D. We approximate this function by a sum $p_o \approx \sum_{l=0}^{L}\sum_{m=-l}^{l} a_{lm}(p_0) Y_l^m$ and describe the function p_0 by the measurements $\{a_{lm}(p_0)\}$. For ease of notation we collect the coefficients $a_{lm}(p_0)$ belonging to the same l in the vector: $A_l(p_0) = (\, a_{-ll} \;\; \ldots \;\; a_{ll} \,)$.

Now we know that if another pattern p is a rotated version of p_0 (i.e. $p = p_0^R$ for some $R \in SO(3)$) then we can find unitary matrices $T_l(R)$ such that $A_l(p) = A_l(p_0^R) = T_l(R)A_l(p_0)$. The spatial relation between the pattern $p = p_0^R$ and the pattern p_0, given by the rotation R, is thus encoded in the transformation matrices $T_l(R)$. In our pattern recognition application we now have the two vectors $A_l(p)$ and $A_l(p_0)$. $A_l(p)$ was computed in the given image and $A_l(p_0)$ is given by our knowledge of the pattern p_0. Given these two vectors we would like to compute the orientation of the pattern p with respect to the given pattern p_0. In the edge- or line-detection example we would like to have an estimation of the orientation of the edge or line at a given point in the image. We must thus compute the rotation R from the knowledge of the vectors $A_l(p)$ and $A_l(p_0)$ and the relation $A_l(p) = T_l(R)A_l(p_0)$. An explicit knowledge of the transformation matrices $T_l(R)$ is thus of great practical importance.

The following derivation of these transformation matrices is based on the traditional treatment of the theory of special functions like the one found in [11].

There one starts with homogeneous, harmonic polynomials defined as follows:

Definition 4.9 1. Let $p \geq 0$ and $H_n(\xi_1, ..., \xi_m)$ be a polynomial of degree n in the m variables ξ_k. H_n is called a *homogeneous* of degree n if:

$$H_n(\lambda \cdot \xi_1, ..., \lambda \cdot \xi_m) = \lambda^n \cdot H_n(\xi_1, ..., \xi_m) \tag{4.64}$$

2. A function f in the m variables ξ_k is called a *harmonic* function if it is a solution to the differential equation:

$$\Lambda^m f = 0 \tag{4.65}$$

where

$$\Lambda^m = \frac{\partial^2}{\partial \xi_1^2} + \ldots + \frac{\partial^2}{\partial \xi_m^2} \tag{4.66}$$

is the Laplacian in m-dimensional space.

Starting from these definitions one can now introduce surface harmonics as follows:

Definition 4.10 Assume H_n is a harmonic, homogeneous polynomial of degree n in the m variables ξ_k. Then we call the function

$$r^{-n} H_n(\xi_1, ..., \xi_m) = H_n(\xi_1/r, ..., \xi_m/r) \tag{4.67}$$

with $r^2 = \xi_1^2 + ... + \xi_m^2$ a *surface harmonic* of degree n.

Surface harmonics are functions defined on the surface of the sphere and the space of all surface harmonics of a fixed degree is invariant under rotations of the sphere. This space of surface harmonics of a fixed degree defines thus a representation space for the group $SO(m)$. For 3-D rotations we find thus again the functions Y_l^m from equation 4.16. For a fixed degree n this space has the dimension $2n + 1$. In the general case we find the following formula for the number of orthogonal surface harmonics of a fixed degree:

Theorem 4.18 If $h(n,p)$ denotes the number of linearly independent surface harmonics of degree n with the $(p+2)$ variables $\xi_1, ..., \xi_{p+2}$ then we have:

$$h(n,p) = (2n+p)\frac{(n+p-1)!}{p!n!} \tag{4.68}$$

In the special cases of 2-D, 3-D and 4-D surface harmonics we find:

$$h(n,0) = 2, \quad h(n,1) = (2n+1), \quad h(n,2) = (n+1)^2 \tag{4.69}$$

For the 4-D surface harmonics we find the following expression (see [11]):

Theorem 4.19 Let $\eta = (\,\eta_1, \quad \eta_2, \quad \eta_3, \quad \eta_4\,)$ be a four-dimensional unit vector and denote the $(n+1)^2$ surface harmonics of degree n by $S_n^{(k,l)}(0 \le k, l \le n)$. Then we have:

1. $$S_n^{(k,l)}(\eta) = (-1)^k(\eta_4 + i\eta_1)^{n-k-l}(\eta_3 + i\eta_2)^{k-l}P_l^{(n-k-l,k-l)}(\eta_2^2 + \eta_3^2 - \eta_1^2 - \eta_4^2) \tag{4.70}$$

   if $n \ge k + l$ and

2. $$S_n^{(k,l)}(\eta) = (-1)^{n-l}(\eta_4 - i\eta_1)^{k+l-n}(\eta_3 - i\eta_2)^{l-k}P_{n-l}^{(k+l-n,l-k)}(\eta_2^2 + \eta_3^2 - \eta_1^2 - \eta_4^2) \tag{4.71}$$

   if $n < k + l$.

where $P_n^{(\alpha,\beta)}(x)$ is the *Jacobi polynomial* defined as:

$$P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m \tag{4.72}$$

Equipped with these notations we can now describe the transformation formula for the surface harmonics:

Theorem 4.20 Assume $R \in SO(3)$ is a rotation with rotation axis described by the unit vector $(\,x_1, \quad x_2, \quad x_3\,)$ and a rotation angle θ with $0 \le \theta \le \pi$. We then describe R by the vector

$$\eta = \eta(R) = (\,x_1 \sin\tfrac{\theta}{2}, \quad x_2 \sin\tfrac{\theta}{2}, \quad x_3 \sin\tfrac{\theta}{2}, \quad \cos\tfrac{\theta}{2}\,).$$

For the surface harmonics we get the following transformation formula:

$$Y_n^k(R(\xi)) = \sum_{l=-n}^{n}(-1)^{k+l}\frac{\binom{2n}{n+k}}{\binom{2n}{n+l}}S_{2n}^{(n+k,n+l)}(\eta)Y_n^l(\xi) \tag{4.73}$$

4.2 Rigid Motions

In the last section we derived in detail the representations of the group of 3-D rotations. In this section we will briefly consider the group of rigid motions in 2-D and 3-D. We recall first the definition of a rigid motion:

Definition 4.11 A *rigid motion* in the real n-dimensional space $\mathbf{R}^n$ is a mapping of the form $X \mapsto RX + T$, where $R \in SO(n)$ is an n-dimensional rotation and T is an n-dimensional translation vector.

We note that:

1. A *rigid motion in* $\mathbf{R}^2$ has the form:
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x\cos\varphi - y\sin\varphi + \xi \\ x\sin\varphi + y\cos\varphi + \eta \end{pmatrix} \tag{4.74}$$
2. A *rigid motion in* $\mathbf{R}^2$ can thus be described by a matrix
$$\begin{pmatrix} \cos\varphi & -\sin\varphi & \xi \\ \sin\varphi & \cos\varphi & \eta \\ 0 & 0 & 1 \end{pmatrix} \tag{4.75}$$
3. The group of rigid motions in n-dimensional space is denoted by $M(n)$.
4. An n-dimensional rigid motion can be described by an $(n+1)\times(n+1)$-matrix of the type:
$$\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \tag{4.76}$$
where R is an element of $SO(n)$ and T is a translation vector from $\mathbf{R}^n$.

As representations space for $M(2)$ we take the space of all infinitely differentiable functions defined on the plane and we define for each motion g the mapping $T(g)$ as:

$$T(g)f(x,y) = f(x\cos\varphi + y\sin\varphi - \xi, -x\sin\varphi + y\cos\varphi - \eta) \tag{4.77}$$

where φ, ξ, η are the parameters describing the motion g.

The infinitesimal operators L_1, L_2, L_3 are defined as:

$$\begin{aligned} (L_1 f)(x,y) &= \frac{\partial}{\partial\xi}(T(g)f)(x,y)|_{\xi=\eta=\varphi=0} \\ (L_2 f)(x,y) &= \frac{\partial}{\partial\eta}(T(g)f)(x,y)|_{\xi=\eta=\varphi=0} \\ (L_3 f)(x,y) &= \frac{\partial}{\partial\varphi}(T(g)f)(x,y)|_{\xi=\eta=\varphi=0} \end{aligned} \tag{4.78}$$

and from the definition of the mapping $T(g)$ in equation 4.77 we find

$$L_1 = -\frac{\partial}{\partial x}$$

$$\begin{aligned} L_2 &= -\frac{\partial}{\partial y} \\ L_3 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{aligned} \tag{4.79}$$

We introduce the new operators L_- and L_+ as

$$\begin{aligned} L_- &= L_1 - iL_2 \\ L_+ &= L_1 + iL_2 \end{aligned} \tag{4.80}$$

and find that L_3, L_+ and L_- act as the following differential operators:

$$\begin{aligned} L_3 &= -\frac{\partial}{\partial \varphi} \\ L_- &= \mathrm{e}^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right) \\ L_+ &= \mathrm{e}^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right). \end{aligned} \tag{4.81}$$

where (r, φ) are the polar coordinates in the plane. Furthermore we have:

$$L_+L_- = L_-L_+ = \Lambda^2 \tag{4.82}$$

where Λ^2 is the 2-D Laplacian.

We will now find the smallest subspaces which are invariant under $M(2)$. Invariant subspaces are invariant under the operators L_i and we construct invariant subspaces for these operators.

We develop an arbitrary function f on the plane into a Fourier series in polar coordinates:

$$f(x,y) = f(r,\varphi) = \sum_m i^{-m}\mathrm{e}^{im\varphi}g_m(r).$$

The space of functions of the form

$$\psi_m(r,\varphi) = i^{-m}\mathrm{e}^{im\varphi}g_m(r) \tag{4.83}$$

is obviously invariant under the action of L_3.

We now start from one fixed element ψ_m and we want to find out what functions we must add to get a function space that is invariant under L_+ and L_- too. From the equations 4.81 we see that

$$\begin{aligned} L_+\psi_m &= \mathrm{e}^{i(m+1)\varphi}h(r) \\ L_-L_+\psi_m &= \mathrm{e}^{im\varphi}\tilde{g}(r) \end{aligned} \tag{4.84}$$

Now we investigate if we can choose g_m in such a way that the L_+, L_- operators do not leave the invariant subspace. We thus want to choose ψ_m such that

$$\begin{aligned} L_+\psi_m &= i\alpha_m\psi_{m+1} \\ L_-\psi_{m+1} &= i\alpha_m\psi_m \\ L_-L_+\psi_m &= -\alpha_m^2\psi_m \end{aligned} \tag{4.85}$$

The operators L_- and L_+ commute and we find therefore $\alpha_m = \alpha_{m+1}$ or $\alpha_m = \alpha$ for all m. Therefore we find that the ψ_m must be solutions of the differential equation:

$$\Lambda^2\psi_m = -\lambda^2\psi_m \tag{4.86}$$

Using the Laplacian in polar coordinates we find that equation 4.86 leads to the following differential equation for the radial functions $g_m(r)$:

$$g_m''(r) - \frac{m^2}{r^2}g_m(r) + \frac{1}{r}g_m'(r) = -\lambda^2 g_m(r) \tag{4.87}$$

From the theory of Bessel functions we find therefore that $g_m(r)$ must be proportional to the Bessel function $J_m(\lambda r)$. This leads to the following theorem:

Theorem 4.21 Define X_λ is the space of all infinitely differentiable functions on the plane that solve the equation 4.86. Each non-zero value of λ leads to an irreducible representation of $M(2)$.

It can be shown that we can choose $|\lambda| = 1$ and that λ and $-\lambda$ define the same subspace. We can therefore select $\lambda = e^{i\beta}$ with $0 \leq \beta < \pi$.

The invariant subspaces under $M(2)$ are thus the subspaces of the functions of the form

$$\psi_m(x, y) = i^{-m}e^{im\varphi}J_m(\lambda r) \tag{4.88}$$

The equations 4.85 become now

$$\begin{aligned} \left(\frac{d}{dz} - \frac{m}{z}\right) J_m(z) &= -J_{m+1}(z) \\ \left(\frac{d}{dz} + \frac{m+1}{z}\right) J_{m+1}(z) &= J_m(z) \end{aligned} \tag{4.89}$$

and a combination of these two equations results in Bessel's differential equation:

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1 - \frac{m^2}{z^2}\right) J_m(z) = 0 \tag{4.90}$$

A complete investigation of the representations of $M(2)$ can be found in [45], chapter IV.

We saw that the invariant subspaces belonging to the 2-D motion group $M(2)$ can be described by the functions $\psi_m(x, y) = i^{-m}e^{im\varphi}J_m(\lambda r)$. In a similar way it is possible to show that the representations of the 3-D motion group $M(3)$ are connected to the functions

$$Y_l^m(\theta, \varphi)J_{l+1/2}(\lambda r) \tag{4.91}$$

where $l = 0, 1, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$.

For an investigation of the representations of the group $M(n)$ of motions in $\mathbf{R}^n$ the reader is referred to [48].

4.3 SL(2,C)

The special linear group $SL(2,\mathbf{C})$ is the group of all 2×2 matrices with complex entries and determinant equal to one. We write an element in $SL(2,\mathbf{C})$ as $\begin{pmatrix} a & c \\ b & d \end{pmatrix}, ac - bd = 1$. If we write an element in $\mathbf{C}^2$ as a vector $(z_1 \quad z_2)$ then we can define the following action of SL(2,C) on $\mathbf{C}^2$:

$$(z_1 \quad z_2) \mapsto (z_1 \quad z_2) \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (az_1 + bz_2 \quad cz_1 + dz_2) \tag{4.92}$$

Denoting the vector $(z_1 \quad z_2)$ by z and the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ by g, equation 4.92 becomes $z \mapsto zg$, where zg is the usual product of a vector and a matrix. From the last equation we find immediately that $z(g_1 g_2) = (zg_1)g_2$. It is thus easy to see that $\mathbf{C}^2$ is a homogeneous space of SL(2,C).

We now construct finite-dimensional representations of SL(2,C) in the usual way: we first select a suitable space of functions on $\mathbf{C}^2$. If f is a function in this space then we define the transformed function f^g as $f^g(z) = f(zg)$. The mapping T defined as

$$T(g)f(z) = f^g(z) = f(zg) \tag{4.93}$$

defines a representation of SL(2,C). From this (infinite-dimensional) representation we construct finite-dimensional representations by restricting T to finite-dimensional, invariant subspaces of the original function space.

In the following we consider functions $\varphi(z_1, z_2, \overline{z_1}, \overline{z_2})$ of the two complex variables z_1 and z_2. We say that φ is homogeneous of degree (n_1, n_2) if

$$\varphi(\gamma z_1, \gamma z_2, \overline{\gamma z_1}, \overline{\gamma z_2}) = \gamma^{n_1} \overline{\gamma}^{n_2} \varphi(z_1, z_2, \overline{z_1}, \overline{z_2}) \tag{4.94}$$

for all complex constants $\gamma \in \mathbf{C}$. The parameters n_1 and n_2 are complex numbers such that the difference $n_1 - n_2$ is an integer. We require this difference to be an integer since we would like $\gamma^{n_1} \overline{\gamma}^{n_2}$ to be a single valued function of γ. We will denote the degree (n_1, n_2) with the symbol χ and we will always assume that the difference between n_1 and n_2 is an integer. In the rest of this section we will write $\varphi(z_1, z_2)$ instead of $\varphi(z_1, z_2, \overline{z_1}, \overline{z_2})$.

For every degree $\chi = (n_1, n_2)$ we define the function space H_χ as the space of functions that satisfy the following two conditions:

- φ is defined for all $(z_1, z_2) \neq (0, 0)$
- $\varphi(z_1, z_2)$ is homogeneous of degree $\chi = (n_1 - 1, n_2 - 1)$
- $\varphi(z_1, z_2)$ is infinitely often differentiable in the variables z_1, z_2 and their complex conjugates.

For every element $\varphi \in H_\chi$ we introduce a new variable z as $z = z_1/z_2$ and we define a new function f by the equations

$$\varphi(z_1, z_2) = z_2^{n_1 - 1} \overline{z_2}^{n_2 - 1} \varphi(\frac{z_1}{z_2}, 1) = z_2^{n_1 - 1} \overline{z_2}^{n_2 - 1} f(\frac{z_1}{z_2}) = z_2^{n_1 - 1} \overline{z_2}^{n_2 - 1} f(z) \tag{4.95}$$

This is a function of the one extended complex variable z and the value of $f(z) = f(z_1/z_2)$ is obviously only a function of the quotient z_1/z_2. The space of these new functions will be denoted by D_χ :

$$D_\chi = \left\{ f : \text{ there is a } \varphi \in H_\chi \text{ such that: } \varphi(z_1, z_2) = z_2^{n_1-1}\overline{z_2}^{n_2-1} f(\frac{z_1}{z_2}) \right\} \tag{4.96}$$

For a function $\varphi \in H_\chi$ and a matrix $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in$ SL(2,C) we define the following transformed function:

$$\varphi^g(z_1, z_2) = \varphi(az_1 + bz_2, cz_1 + dz_2) \tag{4.97}$$

The transformed function φ^g is also in H_χ and we get therefore for each $g \in$ SL(2,C) a linear mapping $T_\chi(g)$ from H_χ to H_χ defined as:

$$(T_\chi(g)\varphi)(z_1, z_2) = \varphi^g(z_1, z_2) \tag{4.98}$$

For two elements $g_1, g_2 \in$ SL(2,C) we find $T_\chi(g_1 g_2) = T_\chi(g_1)T_\chi(g_2)$. The map $g \mapsto T_\chi(g)$ is thus a representation of SL(2,C). The representation space is in this case H_χ. Using the connection between H_χ and D_χ described in the equations 4.95 we see that we can also use D_χ as representation space. In D_χ we get the transformation

$$f^g(z) = (cz + d)^{n_1-1}\overline{(cz + d)}^{n_2-1} f\left(\tfrac{az+b}{cz+d}\right) \tag{4.99}$$

The map $f \mapsto f^g$ will again be denoted by $T_\chi(g)$:

$$(T_\chi(g)f)(z) = f^g(z). \tag{4.100}$$

The spaces D_χ are thus invariant subspaces under the special linear group SL(2,C).

These spaces are usually infinite dimensional but finite dimensional subspaces can be obtained for positive integers n_1 and n_2. We consider the subspaces of H_χ spanned by the polynomials:

$$P^{\nu\mu}_{n_1 n_2}(z_1, z_2) = z_1^{n_1-1-\nu} z_2^{\nu} \overline{z_1}^{n_2-1-\mu} \overline{z_2}^{\mu} (0 \le \nu < n_1; 0 \le \mu < n_2) \tag{4.101}$$

The space spanned by these polynomials has the dimension $n_1 n_2$.

If $P^{\nu\mu}_{n_1 n_2}(z_1, z_2)$ is defined as in equation 4.101 then we find the corresponding function in D_χ as

$$\begin{aligned} P(z) &= z_2^{1-n_1}\overline{z_2}^{1-n_2} P^{\nu\mu}_{n_1 n_2}(z_1, z_2) \\ &= P^{\nu\mu}_{n_1 n_2}(z_1/z_2, 1) = P^{\nu\mu}_{n_1 n_2}(z, 1) \\ &= z^{n_1-1-\nu}\overline{z}^{n_2-1-\mu} \end{aligned}$$

We denote this polynomial again by

$$P^{\nu\mu}_{n_1 n_2}(z) = z^{n_1-1-\nu}\overline{z}^{n_2-1-\mu} \tag{4.102}$$

Under an element $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $P^{\nu\mu}_{n_1 n_2}(z)$ transforms as:

$$P^{\nu\mu}_{n_1 n_2}{}^{g}(z) = (cz+d)^{n_1-1}\overline{(cz+d)}^{n_2-1} P^{\nu\mu}_{n_1 n_2}(z) \tag{4.103}$$

It can be shown that these are essentially the only finite-dimensional invariant subspaces of D_χ. For further information see [37] pages 265 and 298.

By constructing the finite-dimensional, invariant subspaces we have now solved one important problem in the theory of representations of SL(2,C). We will now briefly touch the problem if we can introduce a scalar product in the function space under which the representation becomes unitary.

We first recall the definition of a hermitian functional:

Definition 4.12 Let V be a vector space. A map: $\psi : V \times V \rightarrow \mathbf{C}$ is called a *hermitian functional* if it is linear in the first component and if $\psi(x,y) = \overline{\psi(y,x)}$.

From [14] (page 190) we get the following result for invariant hermitian functionals on D_χ:

Assume $\chi = (n_1, n_2)$ is an arbitrary index (i.e. $n_i \in \mathbf{C}$ and $n_1 - n_2$ is an integer). Then it can be shown that an invariant hermitian functional exists on a D_χ if and only if $n_1 = -\overline{n_2}$ or $n_1 = \overline{n_2}$.

If the n_i are integers then we find that the condition $n_1 = -\overline{n_2}$ implies $n_1 = 0 = n_2$, this leads thus to the trivial representation space of constant image functions. In the second case we find that $n_1 = n_2 = n$. In this case the functional is given by (see [14] section 6.4):

$$(\varphi, \psi) = (-1)^n \frac{i}{2} \int \varphi^{(n,n)}(z)\overline{\psi}(z)\, dz d\overline{z} \tag{4.104}$$

where

$$\varphi^{(n,n)}(z) = \frac{\partial^{2n}\varphi}{\partial z^n \partial \overline{z}^n}.$$

This functional does not define a scalar product (i.e. a positive, Hermitian functional) on D_χ since D_χ is a space of polynomials of degree less then n in z and $\overline{z}$ and $\varphi^{(n,n)}$ is therefore zero on D_χ.

4.4 Exercises

Exercise 4.1 Define the (2-D) Laplace operator Λ^2 as

$$\Lambda^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Show that Λ^2 is in polar coordinates (r, φ) given by:

$$\Lambda^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Exercise 4.2 Define the (3-D) Laplace operator Λ^3 as

$$\Lambda^3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and the spherical Laplace operator Λ as

$$\Lambda = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Show that

$$\Lambda^3 = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right)$$

Exercise 4.3 Prove the statements in theorem 4.1.

Exercise 4.4 Check equation 4.18 for the vector product.

Exercise 4.5 Check the equations 4.20 and 4.23.